University of Ruhuna
Bachelor of Science (General) Degree
Level II (Semester II) Examination
June 2022

Subject: Applied / Industrial Mathematics
Course Unit: AMT/IMT223β (Applied Probability – Information Theory)
Time: Two (02) Hours

Answer **ALL** Questions.
Calculators will be provided.

---

1. (a) State and prove the **Markov's inequality** in the usual notation. [30]
A factory that produces batches of 1000 laptops finds that, on average, two laptops per batch are defective. **Using the above inequality**, estimate the probability that fewer than five laptops in the next batch will be defective. [10]

(b) Define, in the usual notation,

(i) the relative entropy $D(p||q)$ between two probability mass functions p and q. [05]

(ii) the mutual information $I(X;Y)$ between two descrete random variables X and Y. [05]

Prove that for any $x > 0$, $\log_e x \leq x - 1$ with equality iff $x = 1$. [20]

Use the above inequality to show that the relative entropy $D(p||q)$ is non-negative. [20]

Hence, show that the mutual information is also non-negative. [10]

---

2. Consider the joint probability mass function $P(X = x, Y = y)$ of two discrete random variables X and Y given by the following table.

X \ Y	0	1	2
0	1/4	1/8	1/8
1	0	0	1/4
2	0	1/8	1/8

(i) Find the marginal probability mass functions $P(X = x)$ and $P(Y = y)$ of X and Y respectively. Also calculate corresponding entropies $H(X)$ and $H(Y)$. [40]

(ii) **Using the definitions**, find the joint entropy $H(X,Y)$ and mutual information $I(X;Y)$. [40]

(iii) Write down the chain rule for entropy and **hence** find the conditional entropies $H(Y|X)$ and $H(X|Y)$. [20]

---

3. (a) Consider a source alphabet $A = \{a, b, c, d\}$ with the probability distribution $P = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right\}$.
Suppose the prefix code $C = \{0, 10, 110, 111\}$ is recommended.
Calculate the average length $\bar{\ell}$, entropy $H(P)$ and discuss the efficiency of the code. [30]

Continued.

(b) Define the channel capacity C of a **Binary Symmetric Channel (BSC)**. [10]
Write down the channel matrix P and compute the channel capacity C of a **BSC** with input probability distribution

$$\Pr(X = 0) = 0.4, \qquad \Pr(X = 1) = 0.6;$$

and the probability of error (cross over probability) 0.1. [60]

---

4. (a) Define the **differential entropy** of a continuous random variable X. [05]
Find the differential entropy of a normal distribution with mean μ and variance σ^2. [35]
Hence, deduce the differential entropy of a standard normal distribution. [05]

(b) Define the **Kullback-Leibler divergence** $D(f\|g)$ for two probability density functions f and g on the sample space Ω. [05]
Let $f(x) = f(x; \frac{1}{\theta})$ and $g(x) = g(x; \frac{1}{\beta})$ be two exponential distributions where $\theta, \beta > 0$.
Obtain an expression for $D(f\|g)$ in the simplest form. [50]

****