University of Ruhuna

Bachelor of Science (General) Degree

Level III (Semester II) Examination - January 2022

Subject: Mathematics
Course Unit: MAT322β (Complex Variables) . **Time: Two (02) Hours**

Answer All Questions

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1. a) In complex plane, explain what is meant by:
   (i) an open set,
   (ii) a connected set,
   (iii) a domain,
   (iv) a complex function.

   b) Mark the regions denoted by:

   (i) $|z-i| \leq 2,$ (ii) $Re(z) \geq 1,$
   (iii) $|z-2+i| \leq 4$ and $|z| \geq |z-i|,$

   in separate figures.

   c) Let $|z-1| = 2\,|z+2|$, where z represents any complex number. Find the locus of the point $P(x, y)$ in the Argand diagram.

   d) In the usual notion, express the Cauchy's Riemann equations. Hence, show that $f(z) = \bar{z}^2$ is not an entire function.

   e) Find the harmonic conjugate function of $u(x, y) = x^2 - y^2 - y$.

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2. a) For a differentiable function $f(z) = u(x, y) + iv(x, y)$, show that
   (i)
   $$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y},$$
   (ii) $u(x, y)$ and $v(x, y)$ satisfy the Cauchy's Riemann equations.

   b) Show that the function $f(x, y) = e^{(x^2-y^2)}(\cos 2xy + i \sin 2xy)$ satisfies the Cauchy's Riemann equations.

   c) In the usual notation, state the Milne Thomson's method. If $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2$, determine the conjugate function, $v(x, y)$ by using Milne Thomson's method.

   d) Show that the function $f(z) = z^5$ is analytic.
   [You may use the Cauchy's Riemann equations in polar form as $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ and $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$, in the usual notation.]

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3. a) State the Cauchy's integral formula for derivatives. Evaluate the followings:

(i) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$, $C : |z| = 3$.

(ii) $\oint_C \frac{z-3\cos z}{(z-\frac{\pi}{2})^2} dz$, $C : |z| = 2$.

b) (i) Define each of the followings for a complex function

i. a singular point,

ii. a removable singular point,

iii. an essential singular point.

(ii) Find residue of each of the singular points

(i) $f(z) = \frac{\sin z - z}{z^3}$, (ii) $f(z) = \frac{z}{(z-1)(z+3)}$.

c) Obtain all possible Taylor and Laurent series expansions of the function

$$f(z) = \frac{2}{(z+2)(z+1)^2},$$

about $z = 1$.

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4. a) Evaluate $\oint_C f(z)dz$; $C : |z| = 3$ for each of given functions $f(z)$:

(i) $\frac{1}{(z-1)(z-2)}$,

(ii) $\frac{e^{z^2}}{(z-4)^2(z-5)^3}$.

b) In the usual notation, state the Cauchy's Residue Theorem.

Evaluate

(i) $\oint_C \cot z dz$; $C : |z-3| = 1$,

(ii) $\oint_C \frac{e^z - 1}{z(z-1)(z-2)^2} dz$; $C : |z-1| = 2$,

(iii) $\oint_C \frac{1}{\sin\theta - 2\cos\theta + 3} d\theta$: C is a unit circle centered in the origin.
