UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 5 Examination in Engineering: August 2015

Module Number: IS 5310
(Old Curriculum)

Module Name: Complex Analysis and Mathematical Methods

[Three hours]

[Answer **all questions,** each question carries ten marks]

Q1. a) State the Cauchy's integral formula in the usual notation.
Evaluate

$$\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz;\quad C: |z| = 3$$

b) i) Obtain the Taylor's series expansion of $f(z) = \sin z$ upto third order derivative about the point $z = \frac{\pi}{4}$.

ii) Consider the function

$$f(z) = \frac{1}{z^2 - 3z + 2}.$$

Using partial fractions, show that the Laurent's series expansion of $f(z)$ in the region $1 < |z| < 2$ is given by

$$f(z) = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}.$$

Q2. a) Find the image of

i) the square region with vertices $(0,0), (2,0), (2,2), (0,2)$ under the transformation $w = (1+i)z + (2+i)$.

ii) the circle $|z - 3| = 5$ under the mapping $w = \frac{1}{z}$.

Sketch the image on the w-plane in each case.
Describe the nature (that is, the translation, rotation, and expansion or contraction) of the image in part i).

b) State the Cauchy's residue theorem in the usual notation.
Evaluate

$$\int_0^{2\pi} \frac{d\theta}{2\cos\theta + 3}.$$

Q3. a) If $\mathcal{L}\{f(t)\} = F(s)$, then show that $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$, where a is a real constant.

Find $\mathcal{L}\{\sin at\}$ and using the above result and stating any other result you may use, show that

$$\mathcal{L}\left\{\frac{e^{-t}\sin t}{t}\right\} = \cot^{-1}(s+1)$$

b) i) Using partial fractions, find

$$\mathcal{L}^{-1}\left\{\frac{s^2 - 10s + 13}{(s-7)(s^2 - 5s + 6)}\right\}.$$

ii) Apply the convolution theorem to show that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{t\sin 2t}{4}.$$

[Hint: You may use the trigonometric identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ **in integration, if necessary.]**

c) Show, in the usual notation, that

i) $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.

ii) $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$.

Hence, solve the differential equation

$$y'' + 9y = 18t,$$

given that $y(0) = 0 = y\left(\frac{\pi}{2}\right)$

Q4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x; & 0 < x < \pi \\ \pi; & \pi < x < 2\pi \end{cases}$$

with period 2π.

a) Sketch the graph of $f(x)$ in the interval $-2\pi < x < 2\pi$. What can you say about the behaviour of the function at $x = 0$?

b) Calculate the Fourier coefficients and show that the Fourier series of the given function can be written as

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right] - \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right]$$

Hence, deduce that

i) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

ii) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

Q5. a) i) Show that the Fourier transform $F(s)$ of the function

$$f(x) = \begin{cases} x; & \text{if } |x| \le a \\ 0; & \text{if } |x| > a \end{cases}$$

is given by

$$F(s) = \frac{i}{s^2}\sqrt{\frac{2}{\pi}}[\sin sa - as\cos sa]$$

Hence, find the Fourier transform of the function

$$f(x) = \begin{cases} x^2; & \text{if } |x| \le a \\ 0; & \text{if } |x| > a \end{cases}$$

ii) Consider the function $f(x) = e^{-3x}\cos 3x$. Show that the Fourier cosine transform of $f(x)$, that is $F_C[f(x)]$, is given by

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{3(s^2+18)}{s^4+324}\right]$$

[Hint: You may assume that $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$ **]**

b) Show, in the usual notation, that

$$Z[a^n] = \frac{z}{z-a}$$

where a is any real or complex number.

i) Find the inverse Z -transform of

$$F(z) = \frac{z^2+2z}{(z-1)(z^2-5z+6)}$$

by the partial fraction method.

ii) Find the solution of the difference equation

$$y(n+2) - 5y(n+1) + 6y(n) = 4^n;\ y(0) = 0 \text{ and } y(1) = 1$$

by using Z -transforms.