UNIVERSITY OF RUHUNA
25 AUG 2022
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University of Ruhuna

Bachelor of Science General Degree Level I (Semester I) Examination

December 2020

Subject: Applied Mathematics/Industrial Mathematics
Course Unit: AMT111β/IMT111β(Classical Mechanics-I)

Time: Two (02) Hours

Answer all Questions.

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1. (a) A particle P of unit mass moves in the Oxy plane so that its acceleration at time t is $a\omega^2 \sin \omega t \underline{\mathbf{i}} + b\omega^2 \cos \omega t \underline{\mathbf{j}}$, where a, b and ω are positive constants. Initial position and velocity of the particle are $-b\underline{\mathbf{j}}$ and $-\omega a\underline{\mathbf{i}}$ respectively. Find

   (i) velocity of the particle at time t, *[10 marks]*

   (ii) position vector of the particle at time t, *[10 marks]*

   (iii) the Cartesian equation of the path of the particle, *[10 marks]*

   (iv) the work done on the particle when it moves from $t = 0$ to $t = 2$, *[10 marks]*

   (v) torque of the particle at time t. *[10 marks]*

   (b) A particle of mass m is projected vertically upward with an initial speed u_0. The gravity is constant but there is a resistant force mkv^2 at speed v, where k is a constant. Show that

   (i) the maximum height, H, that the particle reaches is given by
   $$2kH = \ln\left(1 + \frac{ku_0^2}{g}\right), \text{ and}$$ *[25 marks]*

   (ii) the speed v_0 when the particle reaches the ground is given by
   $$2kH = \ln\left(\frac{g}{g - kv_0^2}\right).$$ *[25 marks]*

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2. a) Show in the usual notation, that the velocity and acceleration components of a moving particle are given in cylindrical polar coordinates, (r, θ, z), by:

   (i) $\underline{\mathbf{v}} = \dot{r}\hat{\underline{\mathbf{r}}} + r\dot{\theta}\hat{\underline{\theta}} + \dot{z}\hat{z}$ *[20 marks]*

   (ii) $\underline{\mathbf{a}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{\mathbf{r}}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}} + \ddot{z}\hat{z}.$ *[20 marks]*

   b) A particle is projected horizontally at the point $r = a$ with velocity v along the smooth inner surface given by $z = \frac{r^2}{a}$ whose axis is vertically upward. If the particle is at the point (r, θ, z) at time t, referred to cylindrical coordinates.

(i) Show that the kinetic energy, T of the particle is given by
$T = \frac{m}{2}\left(r^2\dot{\theta}^2 + \dot{r}^2 + \dot{z}^2\right)$. *[10 marks]*

(ii) Obtain the equations, $\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 + 2gz = C_1$ and $r^2\dot{\theta} = C_2$, where C_1 and C_2 are constants. *[15 marks]*

(iii) Find C_1 and C_2. *[10 marks]*

(iv) Show that $\left(\frac{a}{4z} + 1\right)\dot{z}^2 + \left(\frac{a}{z} - 1\right)v^2 + 2g(z-a) = 0$. *[10 marks]*

(v) Find an expression for $\ddot{z}$ at $t = 0$. *[10 marks]*

(vi) Show that initially the particle goes up or falls down according to $v^2 \gtrless 2ga$. *[05 marks]*

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3. a) Obtain in the usual notation, the Euler's equations for the motion of a rigid body with one point fixed. *Marks 40*

b) In the usual notations, the moments of inertia and products of inertia of a rigid body with respect to $Oxyz$ coordinate system are $I_{xx} = 3k$, $I_{yy} = 2k$, $I_{zz} = 2k$, $I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0$, $I_{zy} = I_{yz} = k$. Here O is the centre of gravity of the rigid body. Show that the principal moments of inertia of the rigid body are $3k$, $3k$ and k. Let the principal axes correspond to the principal moments of inertia $3k$, $3k$ and k be OX, OY and OZ respectively. The rigid body is free to rotate about its center of gravity, O, without external forces. Initially the body is given an angular velocity $\underline{\omega}_0 = (2\Omega, 0, \Omega)$ with respect to the coordinate system $OXYZ$, where Ω is a constant. *[20 marks]*

(i) Write down the Euler's dynamical equations for this problem. *[10 marks]*

(ii) Show that after time t the angular velocity $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ satisfies the equations

$$\omega_1^2 + \omega_2^2 = 4\Omega^2$$
$$\omega_3 = \Omega.$$

[15 marks]

(iii) Find $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ as functions of time. *[15 marks]*

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4. a) The Lagrange's equations for a dynamical system is given in the usual notation by:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j; \qquad j = 1, 2, ..., n.$$

Deduce the Lagrange's equations for a holonomic conservative dynamical system of the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0; \qquad j = 1, 2, ..., n.$$

[30 marks]

Continued.

b) A uniform tod AB of mass $3m$ and length $2l$ has its middle point O fixed and a particle of mass m is attached at the end B. $OXYZ$ is a coordinate system such that OZ vertically downwards. Initially OB lies along OX and the rod is given an angular velocity $\sqrt{\frac{6g}{l}}$ about OZ. In the subsequent motion OB makes an angle θ with OZ axis and BOZ plane makes an angle ϕ with XOZ plane.

I Show that the kinetic energy, T, of the system is given by
$T = ml^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$. *[20 marks]*

II Assuming OXY plane as the potential energy zero level, show that the potential energy, V, of the system is given by $V = -mgl\cos\theta$. *[05 marks]*

III Write down the Lagrangian of the system. *[05 marks]*

IV Using Lagrange's equations of motion of the system, show that

i. $\dot{\phi}\sin^2\theta = \sqrt{\frac{6g}{l}}$, *[10 marks]*

ii. $\dot{\theta}^2 + \frac{6g}{l}\cot^2\theta + \frac{g}{l}\cos\theta = 0$, and *[20 marks]*

iii. the end B will fall a distance $l(\sqrt{10} - 3)$ in the ensuing motion. *[10 marks]*
