University of Ruhuna

Bachelor of Science General Degree - Level II (Semester I) Examination - February 2022

Subject: Mathematics
Course Unit: MAT 212β (Real Analysis I) **Time: Two (02) Hours**

Answer <u>ALL</u> questions.

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1. a) If the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is given by $S_n = 3 - \frac{n}{2^n}$, find a_n for $n > 1$. Is the series $\sum_{n=1}^{\infty} a_n$ convergent ? Justify your answer. [20 Marks]

   b) Using the comparison test or otherwise show that the series $\sum_{n=1}^{\infty} \frac{1}{n^3 + n^2\cos(n)}$ converges. [15 Marks]

   c) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of positive terms and and suppose that $\frac{a_n}{b_n} \longrightarrow L$, as $n \longrightarrow \infty$,where $0 < L < \infty$. Prove that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges. What can you say about the convergence when $L = 0$?

   [40 Marks]

   d) Determine whether the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$ converges or diverges. [25 Marks]

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2. a) (i) State clearly the integral test for the convergence of the series $\sum_{n=1}^{\infty} a_n$ of positive terms.

   (ii) Test the convergence of the series $\sum_{n=1}^{\infty} ne^{-\frac{n}{2}}$. [40 Marks]

   b) Using a suitable test determine whether the series $\sum_{n=1}^{\infty} \frac{n^{1-3n}}{4^{2n}}$ converges or diverges.

   [20 Marks]

c) Consider the series given by

$$\frac{2\cdot 4}{3\cdot 5}+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}+\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}+\cdots.$$

(i) Write down the general term a_n, $n \geq 1$ of the series.

(ii) Show that the ratio test is inconclusive for determining the convergence of the series $\sum_{n=1}^{\infty} a_n$.

(iii) Test the convergence of the series $\sum_{n=1}^{\infty} a_n$ using Raabe's test. [40 Marks]

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3. a) Let r be a real number. For which values of r is the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^r+2}$ absolutely convergent or conditionally convergent. (No justification is required). [20 Marks]

b) Suppose that the function $f(x)$ has the power series representation

$f(x) = \sum_{n=1}^{\infty} c_n(x-b)^n$, $b \in \mathbb{R}$.

(i) Using the ratio test discuss the radius of convergence and interval of convergence of the power series.

(ii) Show that if the radius of the power series is R then the radius of the power series of $f'(x)$ is also R. [35 Marks]

c) Consider the power series $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$.

(i) Find the radius of convergence R of the series.

(ii) Find the interval of convergence I of the series and determine whether it converges absolutely or conditionally at each point of I. [45 Marks]

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4. a) Let f be a bounded function defined on $[a, b]$ and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition on $[a, b]$.

(i) Define $L(P, f)$ and $U(P, f)$ in the usual notation.

(ii) Define Lower Riemann Integral and Upper Riemann Integral of f on $[a, b]$.

(iii) State the definition f is integrable over $[a, b]$ and $\int_a^b f\,dx$ using the terms in part a(ii) above. [25 Marks]

b) Let $f : [-1, 1] \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x & \text{if } x \in [-1, 0) \\ -x + 1 & \text{if } x \in [0, 1]. \end{cases}$$

(i) Sketch the graph of $f(x)$. **[10 Marks]**

(ii) Using the partition $P = \{-1, -\frac{1}{2}, 0, \frac{1}{3}, 1\}$ of $[-1, 1]$ show that $U(P, f) = \frac{55}{36}$.
. **[25 Marks]**

(iii) Suppose that a sequence of partition of $[-1, 1]$ is given by $P_n = \left\{x_i = -1 + \frac{i}{n}\right\}_{i=0}^{2n}$.
Considering the partition as two parts for $i = 0, 1, \cdots, n$ and $i = n + 1, n + 2, \cdots, 2n$ separately, show that

$$L(P_n, f) = \sum_{i=1}^{2n}(-x_i)\frac{1}{n} + \sum_{i=n+1}^{2n} \frac{1}{n}.$$

and hence obtain an expression for $L(P_n, f)$ in terms of n.
Show that $\int_{-1}^{1} f(x)\, dx = 1$. **[30 Marks]**

c) Let f be a function defined on $[a, b]$. State (without proof) the Riemann Criterion for the Riemann integrability of f on $[a, b]$. **[10 Marks]**
