University of Ruhuna

Bachelor of Science General Degree
(Level III) Semester I Examination - November 2021

Subject: Mathematics
Course unit: MAT313β(Mathematical Statistics-II)

Time: Two (02) Hours

Answer all Questions.

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1. a) Let $X_1, X_2, ..., X_n$ be a random sample of size n from a population with the probability density function

$$f_X(x,\theta) = \begin{cases} \frac{2(\theta - x)}{\theta^2} & , \text{ if } 0 < x < \theta, \\ 0 & , \text{ otherwise} \end{cases}$$

(i) Find $E(X)$

(ii) Find the moment estimator for θ.

(40 marks)

b) Explain the method of maximum likelihood in point estimation.
Let $X_1, X_2, ..., X_n$ be a random sample of size n obtained from a normal distribution with mean μ and the variance σ^2.
Find the maximum likelihood estimators for μ and σ^2.

The annual growth in height of a certain kind of trees follows a normal distribution with unknown mean, μ and unknown variance, σ^2. A random sample of five trees shows the annual growth in height as below: $65cm, 70cm, 80cm, 113cm$ and $155cm$. Find the maximum likelihood estimates for μ and for σ^2 based on these data.

(60 marks)

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2. a) Consider a probability distribution with parameter θ.
Define an unbiased estimator of the parameter θ.

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a population with mean μ and the variance σ^2.

(i) Show that $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ is an unbiased estimator for μ.

(ii) Let $T = \sum_{i=1}^{n} a_i X_i$.

Show that T is an unbiased estimator of μ, if $\sum_{i=1}^{n} a_i = 1$; where a_i are constants for $i = 1, 2, ..., n$

(40 marks)

b) Let $X_1, X_2, ... X_n$ be continuous random variables obtained from a distribution with probability density function $f_X(x, \theta)$.
State Cramer-Rao inequality for the variance of an unbiased estimator of a function $\tau(\theta)$ of the parameter θ, in the usual notation.
When does the equality hold?
Let $X_1, X_2, ..., X_n$ be a random sample of size n obtained from a distribution with probability density function

$$f_X(x, \theta) = \begin{cases} \theta e^{-\theta x} & , \text{if } x > 0, \\ 0 & , \text{otherwise} \end{cases}$$

for parameter θ.
Let T be an unbiased estimator for $\frac{1}{\theta}$.
Show that the Cramer-Rao lower bound for the variance of T is $\frac{1}{n\theta^2}$.
Further show that the equality holds when $T = \frac{\sum_{i=1}^{n} X_i}{n}$.

(60 marks)

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3. Let $X_1, X_2, ..., X_{n_1}$ be a random sample of size n_1 from a normal population with mean μ_1 and known variance σ_1^2. Let $Y_1, Y_2, ..., Y_{n_2}$ be a random sample of size n_2 from a normal population with mean μ_2 and known variance σ_2^2. Assuming that the two samples are independent, explain how you would construct $100(1-\alpha)\%$ confidence interval for $(\mu_1 - \mu_2)$.

(30 marks)

Two large companies A and B produce same kinds of steal rods. It is interested to estimate the difference in mean lengths of steal rods of two companies. A random sample of 25 rods from the company A and a random sample of 36 rods from the company B are independently taken and inspected.
It is shown that the mean lengths of rods of two samples are $250mm$ and $240mm$ respectively.
According to the production reports of last years, the population variance of the lengths of rods of company A is $400mm^2$ and that of company B is $425mm^2$.
Find a 95% confidence interval for the difference between the mean lengths of steal rods of two companies.
State your assumptions clearly.

(70 marks)

4. Explain the following terms used in Statistical Hypothesis testing.

(i) Type I error

(ii) Type II error

(iii) Power function

(30 marks)

A single observation of an exponentially distributed random variable with probability density function given as,

$$f_X(x,\theta) = \begin{cases} \frac{1}{\theta}e^{-\frac{x}{\theta}} & , \text{if } \theta > 0, x > 0, \\ 0 & , \text{otherwise} \end{cases}$$

is used to test the null hypothesis $H_0 : \theta = 10$, against the alternative $H_1 : \theta \neq 10$.

If the null hypothesis is rejected if and only if the observed value is less than 8 and grater than 12, find the probability of Type I error.

Also find the probability of Type II error if $\theta = 12$.

Obtain the power function of the test.

(70 marks)

Continued.