UNIVERSITY OF RUHUNA
FACULTY OF SCIENCE

BACHELOR OF SCIENCE SPECIAL DEGREE IN MATHEMATICS
LEVEL I (SEMESTER I) END SEMESTER EXAMINATION -- OCTOBER, 2021

SUBJECT: MATHEMATICS COURSE UNIT: MSP 3184 - MEASURE THEORY

Answer ALL Questions **Time Allowed: 3 hours**

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1. (a) Let $A \subseteq \mathbb{R}$ be such that $m^*(A) = 0$.
Show that $m^*(B) = m^*(A \cup B) = m^*(B \backslash A)$ for every $B \subseteq \mathbb{R}$. [20 points]

(b) Let (X, Θ, μ) be a measure space. Let $E, F \in \Theta$ be such that $E \subseteq F$.
Is $\mu(F \backslash E) = \mu(F) - \mu(E)$? Justify your answer. [20 points]

(c) Let $X = \{a, b, c, d\}$, where a, b, c and d are all distinct. Let $\mathcal{C} = \{\emptyset, X, \{b\}, \{a, c\}\}$.
Write down $\sigma(\mathcal{C})$, the $\sigma-$ algebra generated by $\mathcal{C}$. [20 points]

(d) Let $f : [0, 1] \to \mathbb{R}$.
Suppose that the set $\{x \in [0, 1] \mid f(x) = \alpha\}$ is measurable for every $\alpha \in \mathbb{R}$.
Is f measurable? Justify your answer. [20 points]

(e) Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3\sqrt{x}; & \text{if } x \text{ is irrational,} \\ 0; & \text{if otherwise.} \end{cases}$$

Find $\int f dm$. [20 points]

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2. (a) Define the *Lebesgue outer measure*, $m^*(E)$, of a subset E of $\mathbb{R}$ and the *Lebesgue measurable subsets* of $\mathbb{R}$. [10+10 points]

(b) Let $A \subseteq \mathbb{R}$ and Let $\{ E_n \}$ be a sequence of mutually disjoint Lebesgue measurable subsets of $\mathbb{R}$.

(i) By *mathematical induction or otherwise*, show that

$$m^* \left(\bigcup_{k=1}^{n} (A \cap E_k) \right) = \sum_{k=1}^{n} m^* (A \cap E_k) \quad \text{for all } n \in \mathbb{Z}^+.$$ [30 points]

[Hint: You may use the fact that $\bigcup_{k=1}^{n} E_k$ is Lebesgue measurable for all $n \in \mathbb{Z}^+$.]

(ii) *Hence,* show that $m^*\left(\bigcup_{n=1}^{\infty}(A\cap E_n)\right)=\sum_{n=1}^{\infty} m^*(A\cap E_n)$. [20 points]

(iii) *Deduce* that $m^*\left(\bigcup_{n=1}^{\infty} E_n\right)=\sum_{n=1}^{\infty} m^*(E_n)$. [10 points]

(iv) Let $\{F_n\}$ be a sequence of subsets of $\mathbb{R}$ such that $F_n\subseteq E_n$ holds for each $n\in\mathbb{Z}^+$. *Using* part (ii), show that $m^*\left(\bigcup_{n=1}^{\infty} F_n\right)=\sum_{n=1}^{\infty} m^*(F_n)$. [20 points]

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3. Define a $\sigma-$ *algebra,* Θ, on a non-empty set X and a *measure,* μ, on a measurable space (X,Θ). [10+10 points]

Let X be an uncountable set.

(a) Define $\Theta:=\{E\subseteq X \mid E \text{ or } E^c \text{ is countable}\}$.

(i) Show that Θ is a $\sigma-$ algebra on X. [25 points]

(ii) Let $S:=\{\{x\}\mid x\in X\}$.
Show that $\Theta=\sigma(S)$, the $\sigma-$ algebra generated by S. [20 points]

(b) Define $\mu:\Theta\to[0,\infty)$ by

$$\mu(E)=\begin{cases}0; & \text{if } E \text{ is countable,}\\ 1; & \text{if } E^c \text{ is countable.}\end{cases}$$

Show that μ is a measure on Θ. [25 points]

Now let $X=\mathbb{R}$. Evaluate (i) $\mu(2\mathbb{Z})$ and (ii) $\mu(\mathbb{R}\backslash 2\mathbb{Z})$. [5+5 points]

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4. (a) Let (X,Θ,μ) be a measure space and $A\in\Theta$ with $\mu(A)=0$.

(i) Let ϕ be a non negative simple measurable function.
Show that $\int_A \phi\, d\mu=0$. [15 points]

(ii) Let f be a non negative measurable function.
Show that $\int_A f d\mu=0$. [15 points]

(b) Let (X,Θ,μ) be a complete measure space and let $f:X\to\mathbb{R}$ be a measurable function. It is given that $f=g$ $\mu-$ almost everywhere.

(i) Show that g is also measurable. [20 points]

(ii) If $f\geq 0$, show that $\int f d\mu=\int g d\mu$. [10 points]

Continued ...

(c) Consider the Lebesgue measure space $([0,1],\mathcal{L},m)$. Let $f:[0,1]\to\mathbb{R}$ be define by $f(x)=x$ and let $g:[0,1]\to\mathbb{R}$ be define by

$$g(x)=\begin{cases}-x; & \text{if } x\in[0,1]\cap\mathbb{Q},\\ x; & \text{if} x\in[0,1]\cap(\mathbb{R}\backslash\mathbb{Q}).\end{cases}$$

(i) Using the definition, show that f is measurable. [15 points]

(ii) Show that g is also measurable. [10 points]

(iii) Evaluate $\int g\,dm$. [15 points]

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5. (a) State clearly the *monotone convergence theorem* and the *Fatou's Lemma*. [10+10 points]

Prove the Fatou's Lemma. [20 points]

(b) Let (X,Θ,μ) be a measure space and let $\{f_n\}$ be a sequence of non-negative measurable functions on X.

(i) If $f_n\leq f$ for all $n\in\mathbb{Z}^+$, where f is an integrable function, show that $\limsup_{n\to\infty}\int f_n d\mu\leq\int\limsup_{n\to\infty}f_n d\mu$. [25 points]

(ii) If f_n converges pointwise to f and $f_n\leq f$ for all $n\in\mathbb{Z}^+$, show that $\lim_{n\to\infty}\int f_n d\mu=\int f d\mu$. [20 points]

(iii) Let f_n converges pointwise to f and $f_n\geq f$ for all $n\in\mathbb{Z}^+$. Is it true that $\lim_{n\to\infty}\int f_n d\mu=\int f d\mu$? justify your answer. [15 points]

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6. State and prove the *Lebesgue Dominated Convergence Theorem*. [10+30 points]

(a) Let (X,Θ,μ) be a measure space and let f be an integrable function. Show that $\mu\left(\{x\in X\mid |f(x)|\geq\alpha\}\right)<\infty$ for all $\alpha>0$. [15 points]

(b) Let (X,Θ,μ) be a measure space and let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty}\int|f_n|\,d\mu<\infty$.
Show that $\sum_{n=1}^{\infty}f_n$ is integrable and $\int\left(\sum_{n=1}^{\infty}f_n\right)d\mu=\sum_{n=1}^{\infty}\int f_n d\mu$. [10+15 points]

(c) Let f be an Lebesgue integrable function on $(0,1)$.
Find the following limit and justify your steps:

$$\lim_{k\to\infty}\int_0^1 k\ln\left(1+\frac{|f(x)|^2}{k^2}\right)dx.$$ [20 points]

[Hint: You may use the elementary inequalities: $\ln(1+t)\leq 2\sqrt{t}$ and $(1+\frac{x}{n})^n\leq e^x$.]

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