University of Ruhuna

Bachelor of Science Special Degree
Level II (Semester I) Examination

January 2021

Subject: Mathematics
Course Unit: MSP 4114 (Ring and Field Theory)
Time: Three (03) Hours

All Questions should be answered

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1. a) Prove that $R = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$ is an integral domain with respect to ordinary addition and multiplication.
Does it form a field? Justify your answer. **[75 marks]**

b) Show that the subset $S = \{0, 2, 4, 6, 8\}$ of the ring $\mathbb{Z}_{10}$ is a subring. Does S have an unity? Justify your answer. **[25 marks]**

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2. a) Define an idempotent element and a nilpotent element in a ring R.
Prove that a non-zero idempotent cannot be nilpotent. **[20 marks]**

b) Define a semiprime ideal.
Consider the ideal $I = \{6n : n \in \mathbb{Z}\}$ in the ring of integers.
Show that I is semiprime.
Is I prime? Justify your answer. **[20 marks]**

c) Let R be a commutative ring with unity. Show that

(i) $R|M$ is an integral domain if and only if M is prime, where M is an ideal of R;

(ii) if an ideal M of R is a maximal ideal of R then $R|M$ is a field.

[You may assume that $R|M$ is a ring.] **[60 marks]**

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3. a) Let f be a homomorphism of a ring R onto a ring R'. Prove that

(i) $\text{Ker}f$ is an ideal of R;

(ii) $R' \cong \dfrac{R}{\text{Ker}f}$. **[55 marks]**

b) Let R be a commutative ring with unity and let I and J be two ideals of R.
Define a map $\phi : R \to \dfrac{R}{I} \times \dfrac{R}{J}$, such that $\phi(x) = (x + I, x + J)$ for all $x \in R$.
If ϕ is onto, show that

(i) $\dfrac{R}{I \cap J} \cong \dfrac{R}{I} \times \dfrac{R}{J}$;

(ii) I and J are comaximal ideal of R. **[45 marks]**

4. a) Define an irreducible polynomial.

Show that the following polynomials are irreducible in corresponding polynomial rings.

(i) $x^2 + x + 2 \in \mathbb{Z}_3[x]$,

(ii) $\dfrac{x^p - 1}{x - 1} \in \mathbb{Q}[x]$. **[30 marks]**

b) (i) Let R be an integral domain. Show that every irreducible element in $R[x]$ is an irreducible polynomial.

(ii) Is the converse of the above statement in (i)is true? Justify your answer. **[40 marks]**

c) Show that $\sqrt{2} + i$ is algebraic over $\mathbb{Q}$. **[10 marks]**

d) Show that the splitting field of $x^4 + 1$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{2}, i)$. **[20 marks]**

,where $i^2 = -1$.