University of Ruhuna

Bachelor of Science Special Degree in Mathematics
Level II(Semester I)Examination - January 2021

Subject: Mathematics
Course Unit: MSP4134 (Functional Analysis) **Time: Three (03) Hours**

Answer ALL questions.

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1. a) (i) Define a metric space (X, d).
   (ii) Let (X, d) be a metric space and $\bar{d} : X \times X \longrightarrow \mathbb{R}$ be defined by

$$\bar{d}(x, y) = \min\{1, d(x, y)\}, \quad x, y \in X.$$

   Prove that $\bar{d}$ is also a metric on X. **[25 Marks]**

   b) (i) Define a convergent sequence and a Cauchy sequence in a metric space.
   (ii) What is meant by saying that a sequence in a metric space is bounded ?
   (iii) Let $\{x_n\}$ be a convergent sequence with limit x in a metric space (X, d). Show that the sequence is bounded. **[35 Marks]**

   c) (i) Define a complete metric space.
   (ii) Prove that, if (X, d) is a complete metric space and Y is a closed subspace of X, then (Y, d) is complete.
   (iii) Let $X = C[0, 2]$ be the set of all continuous functions defined on $[0, 2]$ and the metric d_1 on X be defined by,

$$d_1(f, g) = \int_0^2 |f(x) - g(x)| \, dx, \quad f, g \in X.$$

   Using the sequence of functions $\{f_n\}$ in X defined by

$$f_n(x) = \begin{cases} x^n & \text{if } x \in [0, 1), \\ 1 & \text{if } x \in [1, 2] \end{cases}$$

   show that the metric space (X, d_1) is not complete. **[40 Marks]**

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2. a) (i) Let (X, d) be a metric space. What is meant by saying that a function $f : X \longrightarrow X$ is a contraction ?

(ii) Applying the Banach fixed point theorem show that the integral equation

$$f(x) = \frac{1}{2}x^2 + \int_0^x uf(u)\, du$$

has a unique solution in $C[0, 1]$ under the supremum metric. **[40 Marks]**

b) (i) Let (X, d) and (Y, ρ) be two metric spaces and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined from X into Y. Define the pointwise convergence and uniform convergence of the sequence $\{f_n\}_{n=1}^{\infty}$ to a function f.

(ii) Suppose that a sequence of functions $\{f_n\}$is defined by

$$f_n(x) = x + \frac{1}{n}, \quad x \in \mathbb{R}.$$

Considering the usual metric on $\mathbb{R}$,

i. find the pointwise limit f of the sequence.

ii. show that $\{f_n\}$ converges to f uniformly as $n \longrightarrow \infty$.

iii. show that the sequence $\{f_n^2\}$ converges to f^2 pointwise, but the convergence is not uniform, here $f_n^2(x) = (f_n(x))^2$. **[60 Marks]**

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3. a) (i) Define a normed space .

(ii) Let $X = C[0, 1]$ and let $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_\infty)$ be two normed spaces in the usual notation. Let $\|\cdot\| : X \longrightarrow \mathbb{R}$ be defined by

$$\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}, \quad f \in X.$$

Determine whether $(X, \|\cdot\|)$ is a normed space. **[35 Marks]**

b) (i) Define a Banach space.

(ii) Consider the space X of sequences of real numbers defined by, in the usual notation, $X = \{x \mid x \in l^\infty, x = \{x_i\}, x_i = 0 \text{ for } i > n, n \in \mathbb{N}\}$ with the supremum norm. Using the sequence $\{x_n\}$ in X given by

$$x_n = (1, \frac{1}{2}, \cdots, \frac{1}{n-1}, 0, 0, \cdots),$$

show that $(X, \|\cdot\|)$ is not a Banach space. **[35 Marks]**

c) (i) What is meant by saying that two norms are equivalent?

(ii) Let $X = \mathbb{R}^n$ and let the two norms $\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$ and $\|x\|_\infty = \max\limits_{1 \le i \le n} |x_i|$ be given, where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.
Show that $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$ for all $x \in \mathbb{R}^n$. Are these norms equivalent ? Explain. **[30 Marks]**

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4. a) Let X and Y be two normed spaces and $T : X \longrightarrow Y$ be a linear operator.

(i) State what is meant by saying that T is bounded and continuous.

(ii) Prove that if T is continuous at 0 then it is bounded. **[30 Marks]**

b) Let $(C[-1,1], \|\cdot\|_\infty)$ be the normed space of continuous functions defined on $[-1,1]$ with the supremum norm. Let the operator $T : C[-1.1] \longrightarrow \mathbb{R}$ be defined by

$$T(f) = \int_{-1}^{0} f(t)\ dt - \int_{0}^{1} f(t)\ dt, \quad f \in C[-1,1].$$

(i) Show that T is a linear operator.

(ii) Show that T is bounded and hence deduce that $\|T\| \leq 2$.

(iii) Using the function $f_n : [-1,1] \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$ given by

$$f_n(x) = \begin{cases} -1 & \text{if } x \in [-1, -\frac{1}{n}) \\ nx & \text{if } x \in [-\frac{1}{n}, \frac{1}{n}] \\ 1 & \text{if } x \in (\frac{1}{n}, 1] \end{cases}$$

show that $\|f_n\|_\infty = 1$ and $|Tf_n| = 2 - \frac{1}{n}$. Deduce that $2 \leq \|T\|$ and find $\|T\|$.

[70 Marks]

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5. a) Define an inner product space. **[10 Marks]**

b) Let X be an an inner product space with an inner product $<\cdot,\cdot>$.

(i) If the inner product space X is real, prove that, in the usual notation,

$$4 < x, y >= \|x + y\|^2 - \|x - y\|^2.$$

(ii) Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X converging to x and y respectively. Prove that the sequence $< x_n, y_n >$ converges to $< x, y >$. **[35 Marks]**

c) (i) What is meant by saying that a system $(x_n)_{n\in\mathbb{N}}$ in an inner product space is orthogonal and orthonormal?

(ii) Let $X = C[0,1]$ and the inner product be defined by $< f, g >= \int_0^1 f(t)\bar{g}(t)\ dt$, $f, g \in X$.
Find the value of p such that the functions $f(t) = t$ and $g(t) = 3t - p$ are orthogonal. **[20 Marks]**

d) (i) Let X be a Hilbert space and $\{u_1, u_2, \ldots\}$ be an orthonormal basis of X.

i. Show that for any $x, y \in X$,

$$< x, y >= \sum_{k=1}^{\infty} < x, u_k > \overline{< y, u_k >}.$$

ii. Prove that if $\{\alpha_n\}$ is a sequence of scalars such that the series $\sum_{n=1}^{\infty} |\alpha_n|^2$ convergesnthen $\sum_{n=1}^{\infty} \alpha_n u_n$ converges. **[35 Marks]**
