UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: July 2022

Module Number: IS3302 **Module Name: Complex Analysis and Mathematical Transforms (C 18)**

[Three Hours]

[Answer **all questions**, each question carries twelve marks]

Q1. a) Discuss the continuity of the function $f(z)$ at $z = i$.

$$f(z) = \begin{cases} \frac{z^2+1}{z-i} & ; \ z \neq i \\ 0 & ; \ z = i \end{cases}$$

[3 Marks]

b) Consider the harmonic function $u(x, y) = e^{-x}\cos y + xy$.

i Find a harmonic conjugate of $u(x, y)$.

ii Find the corresponding analytic function $f(z)$.

[3 Marks]

c) In the usual notations, z and w are two complex numbers in Z and W planes respectively.

i Find the image curves of the lines $x = 2$ and $y = 1$ under the mapping, $w = z^2$.

ii Find the angles between the image curves at the point of intersection on the w - plane, where $u > 0$ and $v > 0$.

iii Discuss whether the mapping $w = z^2$ is conformal or not at the point z_0 ; where z_0 is the point of intersection of the lines $x = 2$ and $y = 1$.

[6 Marks]

Q2. a) Find the Maclaurin series of $f(z) = \frac{1}{1-z}$; $|z| < 1$.

[3 Marks]

b) Obtain all possible Laurent's series of the function $f(z) = \frac{1}{z^2-3z+2}$ about $z = 0$.

[4 Marks]

c) Use Cauchy's Residue Theorem to evaluate the following.

$\int_0^{2\pi} \frac{1}{\left(\frac{5}{4}+\sin\theta\right)} d\theta$; C is the unit circle

[5 Marks]

Q3. a) In the usual notations, if the Laplace transform of the function $f(t)$ is given by $L[f(t)] = F(s)$, then show that $L[e^{at}f(t)] = F(s-a)$.

Hence, find the Laplace transform of the followings.

i $e^{-3t}(\cos 4t + 3\sin 4t)$

ii $e^{-t}u(t-a)$

[3 Marks]

b) Using the Laplace transform, solve the initial value problem.

$$\frac{dy}{dt} = y + 3x \quad ; y(0) = 2, x(0) = 1$$
$$\frac{dx}{dt} = 4y - 4e^t \quad ; y'(0) = 3, x'(0) = 2$$

[4 Marks]

c) Consider the following first order linear differential equation that describes the radioactive decay.

$$\frac{dN}{dt} = -\lambda N \quad ; \quad N(0) = N_0$$

where $N = N(t)$ represents the number of undecayed atoms remaining in a sample of a radioactive isotope at time t and λ is the decay constant.
Use Laplace transform to obtain the correct form for radioactive decay $N(t)$.

[2 Marks]

d) Find the inverse Laplace transform of $F(s)$ using Convolution theorem. Where α, k and l values are constants.

$$F(s) = \frac{k}{l} \frac{\alpha}{s^2\left(s^2 + \frac{k}{l}\right)}$$

[3 Marks]

Q4. a) Prove that $Z\{a^n . u(n)\} = \frac{z}{z-a}; |z| > a$.
Hence, find the Z transform of the following sequence.

$$x(n) = \left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$$

where $Z[u(n)] = \frac{z}{z-a} \; ; \; |z| > a$.

[4 Marks]

b) Find the inverse Z transform of the following function using Convolution theorem.

$$F(z) = \frac{z^2}{(z-1)(2z-1)}$$

[3 Marks]

c) Solve the following difference equation using Z transform.

$$f_{k+2} + 6f_{k+1} + 9f_k = 2^k \quad ; \quad f(0) = 0, \; f(1) = 0$$

[5 Marks]

Q5. a) Consider the Fourier Series for a function $f(t)$ of period 2π ;

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

Where,

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt \, dt \; ; \; n = 1,2,3, \ldots \; , \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt \; ; \; n = 1,2,3, \ldots$

i Find the Fourier series expansion of the function $f(t) = t$ of period 2π defined in the interval $(-\pi, \pi)$.

ii Hence, by giving an appropriate value to t, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

[4 Marks]

b) In the usual notations, equations of the Fourier transform and inverse Fourier transform are

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \text{ and } f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \text{ respectively.}$$

i Find the inverse Fourier transform of e^{-x^2}.

(use $\int_0^{\infty} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}$, $\mathcal{F}\{f(at)\} = \frac{1}{a}F\left(\frac{\omega}{a}\right)$, $\mathcal{F}\{f(t - t_0)\} = e^{-i\omega t_0}F(\omega)$, $\mathcal{F}\{e^{i\omega_0 t}f(t)\} = F(\omega - \omega_0)$)

ii Hence, find the Fourier transform of $e^{-2(x-3)^2}$.

[5 Marks]

c) The equation of motion for a damped harmonic oscillator driven by a force $f(t)$, takes the form

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \lambda^2 x = f(t)$$

where the damping constant $k > 0$, and λ^2 is positive constant.
Use Fourier transform to solve the differential equation.

[3 Marks]