# UNIVERSITY OF RUHUNA

## Faculty of Engineering

End-Semester 2 Examination in Engineering: February 2023

**Module Number: IS2401** **Module Name: Linear Algebra & Differential Equations**

**[Three hours]**

[Answer **all questions,** each question carries **12** marks]

**Use separate booklets to answer part A and part B**

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## PART A

Q1. a) A water tank of constant cross-sectional area $10^6\ mm^2$ has a hole of cross-sectional area $10^3\ mm^2$ in its base from which water leaks. Derive a differential equation showing the height $h$ of water in the tank changes with time and hence determine the time taken to empty the tank if it has an depth of $500\ mm$. You may assume the followings.
The velocity with which water emerges from the base is given by $v = \sqrt{2gh}$, and the volume of water leaving the tank per second is $10^3\ \sqrt{2gh}$. In the above, the acceleration due to gravity has been taken as $9.81\ \times 10^3\ mm/s^2$.

[4 Marks]

b) Prove that,

i. $\frac{1}{(1-D)}\{x^n\} = (1 + D + D^2 + D^3 + \cdots + D^n + \cdots)x^n$.

ii. If $F(\alpha) = 0$, then $\frac{1}{F(D)}\{e^{\alpha x}\} = \frac{e^{\alpha x}}{\phi(\alpha)}\ \frac{x^p}{p!}$ for some $p \in \mathbb{Z}^+$, where $F(D) = (D - \alpha)^p \phi(D)$.

[3 Marks]

c) i. If $t = \log x$, then express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$.

ii. Solve

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \sin(\log x).$$

[5 Marks]

Q2. a) What is meant by

i. an ordinary point

ii. a regular singular point of a differential equation of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = F(x).$$

[3 Marks]

b) i. Show that $x = 0$ is a regular singular point of the differential equation

$$7x\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + y = 0.$$

iii.Solve the above differerntial equation about $x = 0$.

[5 Marks]

c) The acceleration of a particle at any time $t \geq 0$ is given by

$$\boldsymbol{a} = \frac{dv}{dt} = 12\cos 2t\,\mathbf{i} - 8\sin 2t\,\mathbf{j} + 16t\,\mathbf{k}.$$

If the velocity $\boldsymbol{v}$ and displacement $\mathbf{r}$ are zero at $t = 0$, find $\boldsymbol{v}$ and $\mathbf{r}$ at any time.

[4 Marks]

Q3. a) State

i. The divergence theorem.

ii. Stockes' theorem.

[3 Marks]

b) Verify stockes' theorem for the function $\mathbf{A} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$, where $C$ is the unit circle in the $xy - plane$ bounding the hemispehere $z = \sqrt{1 - x^2 - y^2}$.

[4 Marks]

c) Let $V$ be the volume bounded by the closed surface $S$. The scalar field $\phi$ and $\varphi$ are acting on the surface $S$. If $\mathbf{n}$ is the outward unit normal vector to the surface $S$ at the point $(x, y, z)$ and $\mathbf{r}$ is the position vector of the point $(x, y, z)$, prove that

i. $\iiint_V (\phi\nabla^2\varphi - \varphi\nabla^2\phi)dv = \iint_S (\phi\nabla\varphi - \varphi\nabla\phi)\, dS.$

ii. $\int_C \phi d\mathbf{r} = \iint_S (\mathbf{n} \times \nabla\phi)ds = \iint_S d\mathbf{S} \times \nabla\phi.$

[5 Marks]

## PART B

Q4. a) If $V$ is a vector space over the scalar field $F$ then show that

i. zero element of $V$ is unique.

ii. the additive inverse of $u \in V$ is unique.

iii. if $kv = \mathbf{0}$, then $k = 0$ or $v = \mathbf{0}$, where $k \in F$ and $v \in V$.

[3 Marks]

b) i. Let $V = M_{2\times2}$. Write down two distinct subsapaces $U$ and $W$ of $V$ such that $U$ and $W$are proper subsets of $V$. Justify your answer.

ii. If $U$ and $W$ are subspaces of vector space $V$ then show that $U \cap W$ is also a subspace of $V$.

iii. Determine whether the set $S = \{(1, -3,1,0), (2, -1, 3,1), (1, -2,2,1), (1, -1,3,2)$ is linearly independent or not.

[5 Marks]

c) Let $V$ be the vector space of all four diamentioanal vectrors. If $U$ and $W$ are subspaces subspaces of $V$ generated by the sets

$$S = \{(1,1,2,1),(0,1,1,-1),(2,3,5,1),(1,0,1,2)\}$$

and

$$T = \{(1,3,4,-1),(1,2,3,0),(2,0,2\ 1),(1,-2,-1,1)\}$$

i. Find bases and dimensions of $U$ and $W$.

ii. Find a basis and the dimention of $U + W$.

iii. Find the dimension of $U \cap W$.

iv. Extend the above bases of $U$ and $W$ to a bases of $V$.

[4 Marks]

Q5. a) Let $T: V \rightarrow U$ be a linear transformation, where $V$ and $U$ are two vector spaces over the field $\mathbb{F}$

i. Briefly explain what is meant by the Nullity and Rank of $T$.

ii. Show that if $\{v_1, v_2, \dots v_n\}$ spans $V$ then $\{T(v_1), T(v_2), \dots T(v_n)\}$ spans Image of $T$.

iii. Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a linear transformation where,

$$T(x,y,z,t) = (x+y+2t, \quad x-2z+t, \quad y+2z+t).$$

Find Bases for $Ker\ (T)$ and $Im\ (T)$. Hence, determine the Nullity and Rank of $T$.

[6 Marks]

b) i. Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and corresponding eigenvectors of the following matrix

$$A = \begin{bmatrix} 5 & 0 & 6 \\ -3 & -1 & -3 \\ -3 & 0 & -4 \end{bmatrix}.$$

ii. Determine whether the matrix $A$ is diagonalizable. Explain your answer.

iii. Write down the matrix whose eigen values are $\lambda_1^2, \lambda_2^2$ and $\lambda_3^2$.

[6 Marks]