UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 2 Examination in Engineering: March 2014

Module Number: IS2401 **Module Name: Linear Algebra & Differential Equations**

[Three hours]

[Answer **all questions**, each question carries **14** marks]

Write your answers for **PART A** and **PART B** in separate booklets

PART A

Q1. a) Solve the initial value problems

i. $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2};$ $y(x_0) = 0$, where $x_0 > 0$.

ii. $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$.

[5.0 Marks]

b) Suppose that an airplane departs from the point $(a,0)$ located due east of its intended destination to an airport located at the origin (0,0). The plane travels with constant speed v_0 relative to the wind, which is blowing due north with constant speed w_0. As indicated in Figure Q1a, we assume that the plane's pilot maintains its heading directly toward the origin. Figure Q1b derives the plane's velocity components relative to the ground as follows.

$$\frac{dx}{dt} = -v_0\cos\theta = -\frac{v_0 x}{\sqrt{x^2 + y^2}}$$

$$\frac{dy}{dt} = -v_0\sin\theta = -\frac{v_0 y}{\sqrt{x^2 + y^2}} + w$$

Hence, the trajectory $y = f(x)$ of the plane satisfies the differential equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{v_0 x}\left(v_0 y - w\sqrt{x^2 + y^2}\right)$$

By setting $k = {}^{w}/_{v_0}$, find the equation of the plane's trajectory for the initial condition $v(a) = 0$.

[5.0 Marks]

c) Suppose $a = 200\ min$, $v_0 = 500 min/h$, $w = 100\ min/h$, and then $k = {}^{w}/_{v_0} = 1/5$; so the plane will succeed in reaching the airport at (0,0) with these values equation of the plane trajectory yields

$$y(x) = 100\left[\left(\frac{x}{200}\right)^{4/5} - \left(\frac{x}{200}\right)^{6/5}\right]$$

If we want to find the maximum amount by which the plane is blown off course during its trip, what is the maximum value of $y(x)$ for $0 \le x \le 200$?

[4.0 Marks]

Q2. a) What is meant by

i. an ordinary point

ii. a regular singular point of a differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = F(x)$$

[2.0 Marks]

b) Show that the infinity is a regular singular points of the differential equation

$$2x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(\frac{1}{x^2} + 1\right)y = 0$$

Solve the above equation about infinity.

[6.0 Marks]

c) Sketch the region R in the xy plane bounded by

i. $y = x^2, x = 2, y = 1$.

ii. Give a physical interpretation to $\iint_R (x^2 + y^2)dxdy$

iii. Evaluate the double integral in ii).

[6.0 Marks]

Q3. a) Calculate the normal of inertia of the spherical shell $x^2 + y^2 + z^2 = 1$ $(z \geq 0)$ with the density μ_0 about the z-axis.

[4.0 Marks]

b) State

i. The divergence theorem

ii. Stockes' theorem.

[2.0 Marks]

Verify the divergence theorem for $A = (2x - z)\mathbf{i} + x^2y\mathbf{j} - xy^2\mathbf{k}$ taken over the region bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

[4.0 Marks]

c) If V is the volume bounded by the closed surface S and $\mathbf{n}$ is the outword unit normal to surface S at the point (x, y, z), prove that

i. $\iiint_V (\nabla \times A)dV = \iint_S (\mathbf{n} \times A)\, dS$

ii. $\int_C \emptyset dr = \iint_S (\mathbf{n} \times \nabla\emptyset)dS = \iint_S ds \times \nabla\emptyset$

[4.0 Marks]

Q4. a) Briefly define the following terms.

i. Subspace of a vector space.
ii. Linear independence of set of vectors.
iii. Basis for a vector space.
iv. Dimension of a vector space. [2.0 Marks]

b) Which of the following subsets is a subspace of R^3? Justify your answer.

i. $W = \{(1, x_1, x_2) | x_1, x_2 \in R\}$
ii. $W = \{(x_1, x_1 - x_2, x_2) | x_1, x_2 \in R\}$ [2.0 Marks]

c) Determine whether the following set of vectors S is linearly independent or linearly dependent. Justify your answer.

i. $S = \{v_1, v_2, v_3\} = \{(1,2,0,1),(2,-1,1,0),(0,5,-1,2)\}$
ii. Let $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ and $S = \{A, A^2, A^3\}$ [3.0 Marks]

d) Show that $W = \{(x_1, x_2, x_3) \in R^3 \mid x_1 + x_2 + x_3 = 0\}$ is a subspace of R^3. Then find a basis for the subspace W. Determine the dimension of W.

[3.0 Marks]

e) Which of the following functions are linear transformations? Justify your answers.

i. $T: R^3 \to R^2$ defined by $T(x,y,z) = (x+y+z+1,\ 0)$
ii. $T: R^2 \to R^2$ defined by $T(x,y) = (\sin x,\ \cos y)$ [4.0 Marks]

Q5. a) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of a matrix A, prove the followings:

i. The inverse matrix A^{-1} has eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$.
ii. The matrix $(A - kI)$ has eigenvalues $(\lambda_1 - k), (\lambda_2 - k), \ldots, (\lambda_n - k)$.
iii. The matrix A^2 has eigenvalues $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$.
iv. If A is upper triangular, then the eigenvalues are exactly the main diagonal entries.

[4.0 Marks]

b) Find the eigenvalues and eigenvectors of A and $(A - 4I)$. Find the trace (A^3).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -2 & 3 \end{bmatrix}$$ [6.0 Marks]

c) Diagonalize the matrix A in part b. [4.0 Marks]

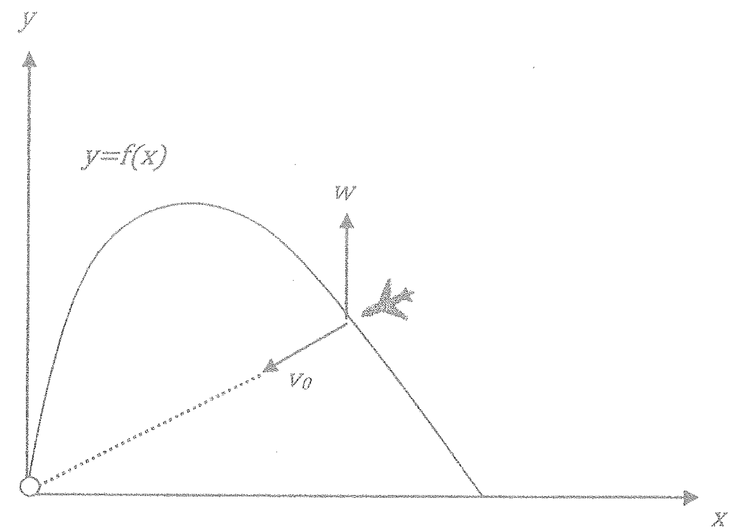

Figure Q1a: The airplane headed for the origin

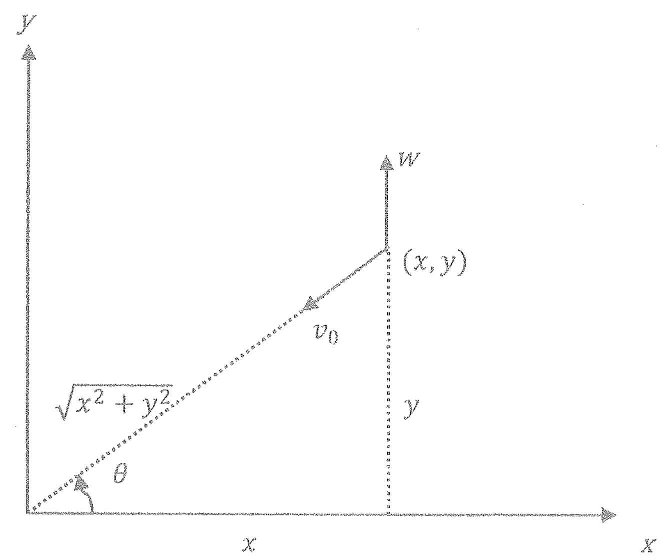

Figure Q1b : The components of the velocity vector of the airplane