UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 5 Examination in Engineering: September 2014

Module Number: IS5310 Module Name: Complex Analysis and Mathematical Methods

[Three hours]

[Answer **all questions**, each question carries **10** marks]

Q1. a) State the Cauchy's integral formula in the usual notation. [1.0 Mark]

Evaluate

(i) $\int_C \frac{z}{z^2-1}\,dz$; $C: |z| = 2,$ [1.5 Marks]

(ii) $\int_C \frac{\exp(2z)}{(z+1)^4}\,dz$; $C: |z| = 3.$ [1.5 Marks]

b)

(i) Obtain the Taylor's series expansion of $\cos z$ up to fifth order derivative about the point $z = \frac{\pi}{2}$. [2.0 Marks]

(ii) Find the Laurent's series expansion of

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

in each of the following regions.

(α) $|z| < 1$

(β) $1 < |z| < 2$

(γ) $|z| > 2$ [4.0 Marks]

Q2. a) Find the image of the circle $|z-2i| = 2$ under the mapping $w = \frac{1}{z}$. Sketch the image on the $w-$plane. [4.0 Marks]

b) State the Cauchy's residue theorem in the usual notation. [1.0 Mark]

Show that

$$\int_0^{2\pi} \frac{d\theta}{\cos\theta + 2\sin\theta + 4} = \frac{2\pi}{\sqrt{11}}.$$

[5.0 Marks]

Q3. a) If $\mathcal{L}\{f(t)\} = F(s)$, then show that $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$ where a is a real constant.

[0.5 Marks]

Find $\mathcal{L}\{\sinh bt\}$ and using the above result obtain $\mathcal{L}\{e^{-at}\sinh bt\}$. [1.5 Marks]

b)

(i) Using partial fractions, show that

$$\mathcal{L}^{-1}\left\{\frac{1+2s}{(s-1)^2(s+2)^2}\right\} = \frac{1}{3}t\exp(t) - \frac{1}{3}t\exp(-2t).$$

[2.5 Marks]

(ii) Using the convolution theorem, find

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}.$$

[1.5 Marks]

c) Show, in the usual notation, that $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$. [0.5 Marks]

Solve the simultaneous equations

$$\frac{dx}{dt} + y = \sin t$$

$$\frac{dy}{dt} + x = \cos t,$$

using Laplace transforms, given that $x(0) = 2$ and $y(0) = 0$. [3.5 Marks]

Q4. a) Find the Fourier series representation for the function $f(x) = \exp(-x)$ in the interval $0 \le x \le 2\pi$. [4.0 Marks]

Hence find

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

when $x = 0$ and $x = \pi$. [1.0 Mark]

b) Expand $f(x) = x$ as

(i) a Fourier cosine series [2.5 Marks]

(ii) a Fourier sine series [2.5 Marks]

in $0 \le x \le \pi$.

Q5. a) Find the Fourier transform of $f(x) = \exp(-a^2x^2)$, where a is a constant.

[2.0 Marks]

Hence deduce that the Fourier transform of $f(x) = \exp\left(-\frac{x^2}{2}\right)$ is $\exp\left(-\frac{\alpha^2}{2}\right)$.

[0.5 Marks]

You may assume that

$$\int_0^{\infty} \exp(-t^2) = \frac{\sqrt{\pi}}{2}.$$

b) The Fourier sine transform of a function $f(x)$ is $\frac{\alpha}{1+\alpha^2}$. Show that

$$f(x) = \sqrt{\frac{\pi}{2}}\exp(-x).$$ [3.0 Marks]

You may assume that

$$\int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha = \frac{\pi}{2}.$$

c) Show that

$$\mathcal{Z}[a^n] = \frac{z}{z-a},$$

where a is any real or complex number. [0.5 Marks]

Solve the following difference equations using $\mathcal{Z}$ −transforms.

(i) $y(n+2) + 5y(n+1) + 4y(n) = 2^n$; $y(0) = 1$ and $y(1) = -4$ [2.0 Marks]

(ii) $x(n+2) - 3x(n+1) - 10x(n) = 0$; $x(0) = 1$ and $x(1) = 0$ [2.0 Marks]