UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: February 2023

Module Number: IS3302 **Module Name: Complex Analysis and Mathematical Transforms (C 18)**

[Three Hours]

[Answer **all questions**, each question carries twelve marks]

Q1. a) Evaluate $\lim_{z\to i}\frac{z^{10}+1}{z^6+1}$.

[2 Marks]

b) Let the function $f(z) = u(x,y) + iv(x,y)$ be an analytic function with the harmonic function $u(x,y) = 3x - 2xy$;

i Find a harmonic conjugate of $u(x,y)$.

ii Find the analytic function $f(z)$.

[4 Marks]

c) Consider the transformation $w = f(z) = z^2$, which lies in the area in the first quadrant of the z-plane bounded by the axes and circles $|z| = 1$ and $|z| = 2$.

i Discuss the transformation in the w-plane.

ii Check whether the transformation is conformal.

[6 Marks]

Q2. a) Find the Maclaurin series of $f(z) = \ln(1+z)$; $|z| < 1$.

[3 Marks]

b) Find the singular points of the following functions.

i $f(z) = \dfrac{3z-2}{(z-1)^2(z+1)(z-4)}$

ii $f(z) = \dfrac{2}{z} + \dfrac{1}{z+i} + \dfrac{3}{(z-i)^4}$

[3 Marks]

c) Consider the function $f(z) = \frac{z+2}{z^2+1}$. If C_1 is a closed contour such that $|z - i| = \frac{1}{2}$ and C_2 is a closed contour such that $|z + i| = 1$, find the integrals below.

i $\oint_{C_1} f(z)\, dz$

ii $\oint_{C_2} f(z)\, dz$

[6 Marks]

Q3. a) In the usual notations, if the Laplace transform of the function $f(t)$ is given by $L[f(t)] = F(s)$, then show that $L[f(t).u(t-a)] = e^{-as}L[f(t+a)]$.
Hence, find the Laplace transform of the followings.

i $f(t) = \begin{cases} e^{2t}\cos 2t \ ; & 2\pi < t < 4\pi \\ 0 \ ; & otherwise \end{cases}$

ii $tU_2(t)$ where $U_2(t) = \begin{cases} 1 \; ; \; t > 2 \\ 0 \; ; \; t < 2 \end{cases}$

[5 Marks]

b) Using the Laplace transform, solve the initial value problem.

$$\frac{dx}{dt} = y + e^t \quad ; \; x(0) = 1$$
$$\frac{dy}{dt} = \sin t - x \quad ; \; y(0) = 0$$

[4 Marks]

c) A RL circuit consisting of a resistance (R) and an inductor (L), connected in a series. Consider that the switch (S) be closed at time $t = 0$ and the input $x(t)$ applied to the circuit is given as $x(t) = V_0.u(t)$. If the current at time t is governed by the differential equation, $L\frac{di(t)}{dt} + Ri(t) = V_0.u(t) \quad ; \; i(0) = 0$, find the current through the inductor $i(t)$ using Laplace transform.

[3 Marks]

Q4. a) Prove that $Z\{a^k.f(k)\} = F(\frac{z}{a})$.

Hence find the Z-transform of the sequence $Z[c^k \sin \alpha k]$, when $k \geq 0$.

[3 Marks]

b) Find the Z-transform of the sequence $\{f(k)\} = \sum_{k=0}^{\infty} 2^k \sum_{k=0}^{\infty} 3^k$ using the Convolution theorem.

[2 Marks]

c) Solve the following difference equation using Z-transform.

$$f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k = U(k) \qquad ; f(0) = f(1) = f(2) = 0$$

[4 Marks]

d) Find the inverse Z-transform of the function $F(z) = \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$ using residue method considering the contour $|z| > 4$.

[3 Marks]

Q5. a) Consider the Fourier Series for a function $f(t)$ of period 2π ;

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \; ;$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt \, dt \; ; \; n = 0,1,2,3, \ldots, \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt \, dt \, ; n = 1,2,3, \ldots$

Consider a function $f(t) = t^2$ of period 2π defined in the interval $(-\pi, \pi)$.

i Sketch a graph of the function $f(t)$ in the interval $-4\pi < t < 4\pi$.

ii Find the Fourier coefficients a_0, a_n and b_n. Then find the Fourier series for $f(t)$ in the interval $-\pi < t < \pi$.

iii By giving an appropriate value to t, show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

[6 Marks]

b) Use Fourier transform and Inverse Fourier transform to solve the differential equation: $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t}U(t)$, where $U(t), = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$.

(Use $\mathcal{F}\{t^n e^{-at}U(t)\} = \frac{n!}{(a+i\omega)^{n+1}}$ where $a > 0$; $\mathcal{F}\left\{\frac{d^n(x)}{dt^n}\right\} = (i\omega)^n F(\omega)$)

[6 Marks]