UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: February 2023

Module Number: IS3301 **Module Name: Complex Analysis and Mathematical Transforms (NC)**

[Three Hours]

[Answer **all questions**, each question carries fourteen marks]

Q1. a) Discuss the continuity of the function $f(z)$ at $z = 0$.

$$f(z) = \begin{cases} \dfrac{Im(z^2)}{|z|^2} & ; \ z \neq 0 \\ 0 & ; \ z = 0 \end{cases}$$

[3 Marks]

b) Let the function $f(z) = u(x,y) + iv(x,y)$ be an analytic function with the harmonic function $u(x,y) = 3x - 2xy$;

i Find a harmonic conjugate of $u(x,y)$.

ii Find $f'(z)$.

[5 Marks]

c) Consider the transformation $w = f(z) = z^2$, which lies in the area in the first quadrant of the z-plane bounded by the axes and circles $|z| = 1$ and $|z| = 2$.

i Discuss the transformation in the w-plane.

ii Check whether the transformation is conformal.

[6 Marks]

Q2. a) Find the Laurent series expansions of the function, $f(z) = \frac{1}{(z-1)(z-2)}$; $z \epsilon \mathbb{C} \setminus \{1,2\}$ around $z = 0$ for $|z| < 1$.

[4 Marks]

b) Find the singular points of the following functions.

i $$f(z) = \frac{3z-2}{(z-1)^2(z+1)(z-4)}$$

ii $$f(z) = \frac{2}{z} + \frac{1}{z+i} + \frac{3}{(z-i)^4}$$

[3 Marks]

c) Use Cauchy's Residue Theorem to evaluate the integral $\oint_C \frac{1}{z^2(z-2)(z-4)} dz$ if C is the rectangle joining the points $(-1,-1)$, $(3,-1)$, $(3,1)$ and $(-1,1)$.

[7 Marks]

Q3. a) In the usual notations, if the Laplace transform of the function $f(t)$ is given by $L[f(t)] = F(s)$, then show that $L[f(t)u(t-a)] = e^{-as}L[f(t+a)]$. Hence, find the Laplace transform of the function $f(t) = tU_2(t)$; where $U_2(t) = \begin{cases} 1, & t > 2 \\ 0, & t < 2 \end{cases}$

[3 Marks]

b) Find the Inverse Laplace Transform of $\frac{S}{S^2+2S+5}$

[3 Marks]

c) Using the Laplace transform, solve the initial value problem.

$$\frac{dx}{dt} = y + e^t \quad ; \; x(0) = 1$$
$$\frac{dy}{dt} = \sin t - x \quad ; \; y(0) = 0$$

[5 Marks]

d) A RL circuit consisting of a resistance (R) and an inductor (L), connected in a series. Consider that the switch (S) be closed at time $t = 0$ and the input $x(t)$ applied to the circuit is given as $x(t) = V_0.u(t)$. If the current at time t is governed by the differential equation, $L\frac{di(t)}{dt} + Ri(t) = V_0.u(t)$; $i(0) = 0$, find the current through the inductor $i(t)$ using Laplace transform.

[3 Marks]

Q4. a) If $x_1(n)$ and $x_2(n)$ are two sequences, then find the Z-transform of their convolution, where $x_1(n) = 4\delta_n + 3\delta_{n-1}$; $x_2(n) = \delta_n - 2\delta_{n-1}$.

[4 Marks]

b) Find the followings.

i $Z\left\{\left(\frac{1}{2}\right)^n \cos n\right\}$

ii $Z^{-1}\left[\frac{Z^{-4}}{Z-1} + Z^{-6} + \frac{Z^{-3}}{Z+0.5}\right]$

[6 Marks]

c) Solve the following difference equation using Z-transform.

$$f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k = U(k) \quad ; f(0) = f(1) = f(2) = 0$$

[4 Marks]

Q5. a) Briefly explain the terms "Periodic Function" and "Sinusoidal Function".

[3 Marks]

b) Consider the Fourier Series for a function $f(t)$ of period 2π ;

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) ;$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt$; $n = 0,1,2,3, \ldots$; $b_n \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt$; $n = 1,2,3, \ldots$

Consider a function $f(t) = t^2$ of period 2π defined in the interval $(-\pi, \pi)$.

i Find the Fourier coefficients a_0, a_n and b_n. Then find the Fourier series for $f(t)$ in the interval $-\pi < t < \pi$.

ii By giving an appropriate value to t, show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

[5 Marks]

c) Use Fourier transform and Inverse Fourier transform to solve the differential equation: $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t}U(t)$, where $U(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$.

(Use $\mathcal{F}\{t^n e^{-at}U(t)\} = \frac{n!}{(a+i\omega)^{n+1}}$, where $a > 0$; $\mathcal{F}\left\{\frac{d^n(x)}{dt^n}\right\} = (i\omega)^n F(\omega)$)

[6 Marks]