University of Ruhuna

Bachelor of Science General Degree
Level II (Semester I) Examination
June 2015

Subject: Mathematics
Course Unit: MAT212β / MPM2123 (Real Analysis I)
Time: Two (02) Hours

Answer Four (04) Questions only

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1. a) State the Comparison Test (1 st type) for infinite series.
Hence

(i) determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ is convergent or divergent;

(ii) show that the series $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ is divergent.

b) Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have positive terms and $\frac{a_n}{b_n} \to l$ as $n \to \infty$, where $l \neq 0$ is finite.
Show that

(i) $\sum_{n=1}^{\infty} a_n$ is convergent if $\sum_{n=1}^{\infty} b_n$ is convergent.

(ii) $\sum_{n=1}^{\infty} a_n$ is divergent if $\sum_{n=1}^{\infty} b_n$ is divergent.

Use the above test to determine whether the series $\sum_{n=1}^{\infty} \left[(n^3+1)^{\frac{1}{3}} - n\right]$ is convergent or divergent.

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2. a) Show that the geometric series $1 + r + r^2 + ...$ with positive terms converges if $r < 1$ and diverges if $r \geq 1$.

b) Use Cauchy's Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, if $p > 1$ and diverges if $p \leq 1$.

c) Does the series $\sum_{n=1}^{\infty} (\frac{1}{n^5} + 1.5^n)$ converge? Justify your answer.

d) For which positive values of k can the Ratio Test be used to prove that $\sum_{n=1}^{\infty} \frac{k^n n!}{n^n}$ is convergent. Justify your answer.

Continued.

3. Let $\sum_{n=1}^{\infty} a_n$ be a positive term series such that $\frac{a_n}{a_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^p}$, where $\alpha > 0$, $p > 1$ and $\{\gamma_n\}$ is a bounded sequence.
Show that

a) $\sum_{n=1}^{\infty} a_n$ converges if $\alpha > 1$ and diverges if $\alpha < 1$, whatever β may be.

b) for $\alpha = 1$, $\sum_{n=1}^{\infty} a_n$ converges if $\beta > 1$ and diverges if $\beta < 1$.

Discuss the convergence of the series
$$\sum_{n=1}^{\infty} \frac{(n!)^2 x^{2n+1}}{(2n)!}, \quad x > 0.$$

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4. a) Define the absolute convergence and conditional convergence of a given series $\sum_{n=1}^{\infty} a_n$.
Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

(i) $\frac{(-1)^n}{n^4 + 7}$

(ii) $\frac{(-1)^n}{2n}$

b) Explain why the alternating series test cannot be used to decide the convergence or divergence of the series
$$\sum_{n=1}^{\infty} (-1)^{n-1}\left(3 - \frac{1}{n}\right).$$

c) State the Abel's Test.
Use Abel's Test to verify the convergence of the series
$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^3}.$$

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5. a) Let f be a bounded function on $[a, b]$. Prove, in the usual notation , that

(i) $L(P, f) \leq U(P, f)$;

(ii) $\underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx$.

[You may assume that $L(P, f) \leq U(P^* f)$, where P^* is a refinement of the partition P of $[a, b]$.]

Continued.

b) Let $f(x) = x$, $x \in [0,1]$ and $P_n = \left\{[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \cdots, [\frac{i-1}{n}, \frac{i}{n}], \cdots, [1 - \frac{1}{n}, 1]\right\}$ be the standard partition of $[0,1]$. By using part a)(ii), show that f is integrable and $\int_0^1 f\,dx = \frac{1}{2}$.

c) Let $f(x)$ be defined on [0,1] as follows:

$$f(x) = \begin{cases} (1-x^2)^{\frac{1}{2}} & ; \quad x \text{ is rational.} \\ 1 - x & ; \quad x \text{ is irrational.} \end{cases}$$

Find the upper and lower Riemann integrals for the function f.
Is f Riemann integrable on $[0,1]$? Justify your answer.

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6. a) Let f be a bounded function on the interval $[a,b]$, and P and P^* be partitions of $[a,b]$, where P^* is a refinement of P. Prove, in the usual notation, that $L(P,f) \leq L(P^*,f)$.
What is the relation between $U(P,f)$ and $U(P^*,f)$?

b) (i) Let f be a bounded function on $[a,b]$. Prove , in the usual notation, that the function f is Riemann integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a Riemann partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

(ii) Let $f : [3,6] \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 2\,; & 3 \leq x < 4 \\ 1\,; & x = 4 \\ 4\,; & 4 < x \leq 6. \end{cases}$$

For the partition $P_k = \{3, 4-k, 4+k, 6\}$, where $0 < k < 1$, use the Riemann's criterion in b)(i) to determine the integrability of the function f.
