University of Ruhuna

Bachelor of Science General Degree
Level III (Semester I) Examination

June 2015

Subject: Mathematics
Course Unit: MAT 311β /MPM 3113 (Group Theory)
Time: Two (02) Hours

Answer Four (04) Questions only

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1. a) Show that
$$G = \left\{ \begin{pmatrix} p & q \\ 0 & r \end{pmatrix} \mid p, q, r \in \mathbb{R},\ pr \neq 0 \right\}$$
forms a group under matrix multiplication.

b) Prove that a non-empty subset H of a group G is a normal subgroup of G if and only if $(gx)(gy)^{-1} \in H$ for all $x, y \in H,\ g \in G$.

c) Using the above part b) show that $N = \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{R} \right\}$ is a normal subgroup of G.

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2. a) Let $G = < a >$ be a cyclic group of order n. Show that $G = < a^m >$ if and only if $\gcd(m, n) = 1$.

b) Consider the set $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(a, b) : a \in \mathbb{Z}_2,\ b \in \mathbb{Z}_3\}$.

(i) Write down the elements of $\mathbb{Z}_2 \times \mathbb{Z}_3$.

(ii) Show that $\mathbb{Z}_2 \times \mathbb{Z}_3$ forms a group under the operation $\oplus$ defined by $(a, b) \oplus (c, d) = (a \oplus_2 c\ ,\ b \oplus_3 d)$ for any $(a, b), (c, d) \in \mathbb{Z}_2 \times \mathbb{Z}_3$.

(iii) Does $\mathbb{Z}_2 \times \mathbb{Z}_3$ form an abelian group? Justify your answer.

(iv) Show that $G =< (1, 1) >$.

(v) Using part a) find the other generators of the cyclic group $\mathbb{Z}_2 \times \mathbb{Z}_3$.

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3. a) Let $\alpha = (6217)(2413)$ be a permutation defined on the set $X = \{1, 2, 3, 4, 5, 6, 7\}$.

(i) Express α as a product of disjoint cycles and product of transpositions.

(ii) Is α an odd permutation? Justify your answer.

b) Let $p = (135)$, $q = (241)$ and $r = (2354)$ be permutations defined on the set $X = \{1, 2, 3, 4, 5\}$. Find $\tau = p^2 q^{-1} r$ and $o(\tau)$.

c) Let H and K be two subgroups of a group G.
Show that

(i) $H \cap K$ is a subgroup of G;

(ii) If H and K are both normal in G then $H \cap K$ is normal in G.

Continued.

4. a) Let a and b be arbitrary distinct element of a group G and, H be any subgroup of G.
Show that

(i) $Ha = H \Leftrightarrow a \in H$

(ii) $Ha = Hb \Leftrightarrow ab^{-1} \in H$

b) Show that any two right (left) cosets of a subgroup are either disjoint or identical.

c) A subgroup H of a group G is called a normal subgroup if $xH = Hx$ for every element $x \in G$.
Let $H = \{0, 2, 4\}$ be a subgroup of the group $G = (\{0, 1, 2, 3, 4, 5\}, \oplus_6)$.

(i) List all the left and right cosets of H in G.

(ii) Hence determine whether H is normal in G.

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5. a) Let G and G' be two groups and $f : G \to G'$ be a homomorphism.
Define the kernel of f (Ker$f$).
Prove that

(i) Kerf is a normal subgroup of G;

(ii) f is one-one if and only if Ker $f = \{e\}$, where e is the identity of G.

b) Let $(G, *)$ and $(G', \circ)$ be two groups. Define what is meant by an isomorphism $f : G \to G'$.
Let $(\mathbb{R} \setminus \{-1\}, *)$ and $(\mathbb{R} \setminus \{0\}, .)$ be two groups. The operation $'*'$ is defined by $a * b = a + b + ab$ for all $a, b \in \mathbb{R} \setminus \{-1\}$ and $'.'$ is the usual multiplication.

(i) Define a suitable map $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{-1\}$.
Hence show that $(\mathbb{R} \setminus \{0\}, .) \cong (\mathbb{R} \setminus \{-1\}, *)$

(ii) Find kernel of f.

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6. Prove or disprove the following statements.

a) Let H and K be two subgroups of a group G. HK is a subgroup of G if $HK = KH$.

b) Let H be any non-empty subset of a group G such that $H^{-1} = H$, where $H^{-1} = \{h^{-1} : h \in H\}$. Then H is a subgroup of G.

c) If a is a generator of a cyclic group G then a^{-1} is also a generator.

d) $(\mathbb{Q}, +) \cong (\mathbb{Q}^*, .)$, where $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$.

e) Let G' be a commutator subgroup of a group G. Then G' is normal in G.
