University of Ruhuna

Bachelor of Science General Degree Level III
(First Semester Examination)

June/July 2015

Subject: Mathematics
Course Unit: MAT312β/MPM3123(Real Analysis III)

Time: Two (02) Hours

Answer 04 Questions only.

---

1. a) In the usual notation, prove the Cauchy-Schwarz inequality, $|\underline{\mathbf{x}}.\underline{\mathbf{y}}| \leq \|\underline{\mathbf{x}}\|\|\underline{\mathbf{y}}\|$, where $\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$, $\in \mathbb{R}^n$. Using Cauchy-Schwarz inequality prove

   (i) the triangle inequality, $\|\underline{\mathbf{x}}+\underline{\mathbf{y}}\| \leq \|\underline{\mathbf{x}}\| + \|\underline{\mathbf{y}}\|$,

   (ii) the pythagorian theorem, $\|\underline{\mathbf{x}}+\underline{\mathbf{y}}\|^2 = \|\underline{\mathbf{x}}\|^2 + \|\underline{\mathbf{y}}\|^2$ and

   (iii) if x, y and z are three positive real numbers such that $x+y+z \leq 3$ then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 3$.

   b) Let $\underline{\mathbf{x}}_0 \in \mathbb{R}^n$, and $r > 0$. Define the following terms:

   (i) Open ball in $\mathbb{R}^n$ with center $\underline{\mathbf{x}}_0$ and radius r.

   (ii) closed ball in $\mathbb{R}^n$ with center $\underline{\mathbf{x}}_0$ and radius r.

   c) Explain whether each of the the following set is open or not.

   (i) $A = \{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 < 2\}$ (ii) $B = \{(x,y) \in \mathbb{R}^2 \ : \ 1 \leq x^2+y^2 < 2\}$

   (iii) $C = \displaystyle\bigcup_{n=1}^{\infty}(-n,n)$ (iv) $D = \displaystyle\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$

   d) Explain whether each of the the following set is closed or not.

   (i) $E = \{\underline{\mathbf{x}} \in \mathbb{R}^n \ : \ \|\underline{\mathbf{x}}\| \geq 2\}$ (ii)$F = \{\underline{\mathbf{x}} \in \mathbb{R}^n \ : \ 1 \leq \|\underline{\mathbf{x}}\| \leq 4\}$

---

2. a) Find each of the following limit if it exists or show that the limit does not exist.

   (i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ (ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y^2}{x^2+y^2}$ (iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2+x^4}{(x^2+y^2)^{3/2}}$

   (iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x}$ (v) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\ln(x^2+y^2+1)}{x^2+y^2}$

   b) (i) State the (ϵ,δ) definition for the continuity of the function $f:\mathbb{R}^n \to \mathbb{R}$ at the point $\underline{\mathbf{x}}_0 \in \mathbb{R}^n$.

(ii) Using the (ϵ, δ) approach to limits, show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

has limit 0 at the origin. Explain the continuity of f at the origin.

c) Discuss the continuty of the following function

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Is $f(x,y)$ differentiable at $(0,0)$? Explain your answer.

---

3. a) (i) Let $f : \mathbb{R}^n \to \mathbb{R}$. Define the directional derivative of f at $\underline{\mathbf{a}} \in \mathbb{R}^n$ in the direction of $\underline{\mathbf{u}}$

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. In the usual notation show that $D_{\underline{\mathbf{u}}}f(\underline{\mathbf{a}}) = \nabla f(\underline{\mathbf{a}}).\underline{\mathbf{u}}$.

b) (i) Let $f(x,y) = 3x^2 - y^2 + 4x$ and $\underline{\mathbf{u}} = \frac{\sqrt{3}}{2}\underline{\mathbf{i}} + \frac{1}{2}\underline{\mathbf{j}}$ be a unit vector. Evaluate $D_{\mathbf{u}}f(0,0)$ by using the definition of the directional derivative.

(ii) Let $f(x,y,z) = 3x^2 + xy - 2y^2 - yz + z^2$. Using the result in part a(ii), find the directional derivative of f at $(1,-2,-1)$ in the direction of $2\underline{\mathbf{i}} - 2\underline{\mathbf{j}} - \underline{\mathbf{k}}$.

c) Consider the function $f(x,y) = \dfrac{x^2}{2} + \dfrac{y^2}{2}$.

(i) Find the direction in which $f(x,y)$ increases most rapidly at $(1,1)$.

(ii) Find the direction in which the directional derivative of $f(x,y)$ at $(1,1)$ is 0.

---

4. a) Let $u = x^2 + 2xy + y^2$, $x = t\cos t$, $y = t\sin t$. Find $\dfrac{du}{dt}$ by two methods:

(i) using the chain rule,

(ii) expressing u in terms of t before differentiating.

b) Three ants A, B, C crawl along the positive x, y and z axes respectively. A and B are crawling at a constant speed of 1cm/s, C is crawling at a constant speed 3cm/s and they are all travelling away from the origin.

(i) If $A \equiv (x,0,0)$, $B \equiv (0,y,0)$ and $C \equiv (0,0,z)$ then show that the area of the traingle ABC is given by $\frac{1}{2}\sqrt{x^2y^2 + y^2z^2 + z^2x^2}$.

(ii) Find the rate change of the area of the triangle ABC when A is 2cm away from the origin while B and C are 1cm away from the origin.

c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function at (x_0, y_0). Show that

(i) the tangent plane to the surface $f(x, y, z) = 0$ at the point (x_0, y_0, z_0) is given by the equation $f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$ and

(ii) the normal line to the surface at (x_0, y_0, z_0) is given by the equation $\dfrac{(x - x_0)}{f_x(x_0, y_0, z_0)} = \dfrac{(y - y_0)}{f_y(x_0, y_0, z_0)} = \dfrac{(z - z_0)}{f_z(x_0, y_0, z_0)}$ in symetric form

d) For the surface given by the equation $z = x^2 - 3y^2 + xy$, find the equations of the
(i) tangent plane and
(ii) normal line
at the point $(1, 1, -1)$ on the surface.

---

5. a) (i) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function defined by $f(x_1, x_2, ..., x_n) = f(\underline{\mathbf{x}}) = (f_1(\underline{\mathbf{x}}), f_2(\underline{\mathbf{x}})..., f_m(\underline{\mathbf{x}}))$. Write down the Jacobian matrix of f at the point $\underline{\mathbf{x}}$.

(ii) A function $f : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by the equation $f(x, y) = (\sin x \cos y, \sin x \sin y, \cos x \cos y)$. Determine the Jacobian matrix $Df(x, y)$ of f at the point (x, y).

b) Assume that f is a differentiable function at each point $(x, y) \in \mathbb{R}^2$. Let g_1, g_2 be defined on $\mathbb{R}^3$ by the equations $g_1(x, y, z) = x^2 + y^2 + z^2$, $g_2(x, y, z) = x + y + z$, and let g be the vector valued function given by $(g_1(x, y, z), g_2(x, y, z))$. Let h be the composite function $h = f \circ g$. In the usual notaion, show that

$$\|\nabla h\| = 4(D_1 f)^2 g_1 + 4(D_1 f)(D_2 f) g_2 + 3(D_2 f)^2.$$

c) Let f be the function defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq 0 \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

d) Suppose that $\sin xy + \sin yz + \sin xz = 1$, defines the variable z as a function of x and y. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

---

6. a) Find and classify the extreme values (if any) of the following function: $f(x, y) = 6x - 4y - x^2 - 2y^2$

b) Prove that if a rectangular box without a top is to be made from a given amount of material, then the box will have the largest possible volume if it has a square base and an altitude whose length is half that of the base.

c) (i) Show that $\displaystyle\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dx dy = \pi$

(ii) Evaluate $\displaystyle\iint_D (x + 2y) dx dy$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.
