University of Ruhuna

Bachelor of Science General Degree
(Level III) Semester I Examination - June/July 2015

Subject : Mathematics
Course unit: MAT313β /MMS3113 (Mathematical Statistics II)

Time :Two (02) Hours

Answer four (04) Questions only

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1. Let $\hat{\theta}$ be a point estimator for parameter θ.
Explain what is meant by saying that, $\hat{\theta}$ is an unbiased estimator for θ.

a) Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a population with mean μ and the variance σ^2.

(i) Show that $\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$ is an unbiased estimator for μ and $S^2 = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}{(n-1)}$ is an unbiased estimator for σ^2.

(ii) Let $T = \sum_{i=1}^{n} a_i Y_i$.

Show that T is an unbiased estimator of μ, if $\sum_{i=1}^{n} a_i = 1$; where $a_1, a_2, ..., a_n$ are constants.

b) Let $X_1, X_2, ..., X_n$ be a random sample of size n from a population with the mean μ and the variance σ_1^2 and $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a population with the mean μ and the variance σ_2^2.
If the two samples are independent,

(i) show that $\omega\bar{X} + (1-\omega)\bar{Y}$ is an unbiased estimator of μ ; where $0 \leq \omega \leq 1$,
$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ and $\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$.

(ii) show that the variance of this estimator is a minimum when $\omega = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$.
Find the value of the minimum variance.

Continued.

2. a) Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a population with the probability density function

$$f_Y(y,\theta) = \begin{cases} \frac{2}{\theta} - \frac{2y}{\theta^2} & , \text{if } 0 < y < \theta, \\ 0 & , \text{otherwise} \end{cases}$$

Find

(i) $E(Y)$

(ii) the moment estimator of θ.

b) Explain the method of maximum likelihood estimation.

(i) Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a population with probability density function

$$f_Y(y,\alpha) = \begin{cases} 2\alpha y e^{-\alpha y^2} & , \text{if } y > 0, \alpha > 0, \\ 0 & , \text{otherwise} \end{cases}$$

Find the maximum likelihood estimator of α.

(ii) Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n obtained from a Poisson distribution with probability mass function

$Pr(Y = y) = \frac{e^{-\lambda}\lambda^y}{y!}$, y=0,1,2,......

Find the maximum likelihood estimator of λ.

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3. a) Define the property consistent, of an estimator $\hat{\theta}_n$ of the parameter θ, used in parametric point estimation.

Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a distribution with probability density function,

$$f_Y(y,\theta) = \begin{cases} \theta y^{\theta-1} & , \text{if } 0 < y < 1, \quad \theta > 0, \\ 0 & , \text{otherwise} \end{cases}$$

Show that $\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$, is a consistent estimator of $\frac{\theta}{\theta+1}$.

b) State Cramer-Rao inequality for the variance of an unbiased estimator of a function $\tau(\theta)$ of a parameter θ, in the usual notation.

When does the equality hold?

Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n obtained from a normal distribution with known mean μ and unknown variance σ^2.

Let $T = \frac{\sum_{i=1}^{n}(Y_i - \mu)^2}{n}$.

Continued.

Show that T is a minimum variance unbiased estimator of σ^2.
Obtain Cramer-Rao lower bound for the variance of T.

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4. (a) State Neymass factorization theorem for a sufficient statistic.
Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a population with probability density function $f_Y(y, \theta) = a(\theta)b(y)exp^{[c(\theta)d(y)]}$.

Show that $\sum_{i=1}^{n} d(Y_i)$ is a sufficient statistic for parameter θ.

(b) Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n obtained from a Poisson distribution with probability mass function
$Pr(Y = y) = \frac{e^{-\lambda}\lambda^y}{y!}$, y=0,1,2,...... ; where λ is an unknown parameter.
Show that

(i) $S = \sum_{i=1}^{n} Y_i$ is a sufficient statistic for λ,

(ii) $\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$ is an unbiased estimator for λ and

(iii) $\bar{Y}$ is the minimum variance unbiased estimator for λ.

In the usual notation, state the theorem you used.

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5. Let $X_1, X_2, ..., X_{n_1}$ be a random sample of size n_1 from a normal distribution with unknown mean μ_1 and the unknown variance σ^2. Let $Y_1, Y_2, ..., Y_{n_2}$ be a random sample of size n_2 from a normal distribution with unknown mean μ_2 and the same unknown variance σ^2. Assuming that the two samples are independent and the sample sizes are small, explain how you would construct $100(1-\alpha)\%$ confidence interval for $(\mu_1 - \mu_2)$.

It is needed to compare two kind of fertilizers used in 'Rambutan' plantation. In the plantation some plants were treated with the fertilizer A and the rest were with the fertilizer B. From the plantation, a random sample of five plants treated with fertilizer A and a random sample of five plants treated with fertilizer B, were selected and the harvest was recorded as follows.

No. of fruits of plants treated with fertilizer A	850	833	848	796	803
No. of fruits of plants treated with fertilizer B	771	789	792	827	786

Construct 95% confidence interval for the difference between the mean number of fruits of plants treated with fertilizer A and fertilizer B.
State your assumptions clearly.

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6. a) Explain the following terms used in Statistical Hypothesis testing.

(i) Type I error

(ii) Type II error

(iii) Power function

b) Let Y be a random variable of a distribution whose probability density function is,

$$f_Y(y,\theta) = \begin{cases} \frac{1}{\theta}e^{\frac{-y}{\theta}} & , \text{if } y > 0, \quad \theta > 0, \\ 0 & , \text{otherwise} \end{cases}$$

A single observation from this distribution is used to test the null hypothesis $H_0 : \theta = 10$, against the alternative $H_1 : \theta \neq 10$.

If the null hypothesis is rejected when the observed value is less than 8 and greater than 12, find the probability of Type I error.

Also find the probability of Type II error if $\theta = 12$.

Obtain the power function of the test.
