University of Ruhuna

Bachelor of Science Special Degree in Mathematics- Level I (Semester I Examination)

June/July 2015

Course Unit:MSP3144(Mathematical Methods in Physics and Engineering)

Time: Three (03) Hours

Answer **six (06)** Questions selecting **two (02)** from each section

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Section A

1. a) Define $\mathcal{L}\{F(t)\}$, the Laplace transform of a function $F(t)$, denoted by $f(s)$.

b) If $\mathcal{L}\{F(t)\} = f(s)$ then show that

(i) $\mathcal{L}\{e^{at}F(t)\} = f(s-a),\ s > a$ (ii) $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2 - a^2},\ s > |a|$

(iii) $\mathcal{L}\{tF(t)\} = -\dfrac{d}{ds}f(s)$ (iv) $\mathcal{L}\{\dfrac{F(t)}{t}\} = \displaystyle\int_s^\infty f(u)du$

c) Find the following Laplace transformations:

(i) $\mathcal{L}\{(2e^{3t}\sin 4t\}$, (ii) $\mathcal{L}\{t\sinh 4t\}$,

(iii) $\mathcal{L}\left\{\dfrac{\sinh t}{t}\right\}$, (iv) $\mathcal{L}\left\{\dfrac{e^{-2t} - e^{-3t}}{t}\right\}$,

(v) $\mathcal{L}\left\{\displaystyle\int_0^t \frac{\sin u}{u} du\right\}$.

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2. a) Define the inverse Laplace transform $\mathcal{L}^{-1}\{f(s)\}$ of $f(s)$.

b) Show, in the usual notation, that

$\mathcal{L}^{-1}\{f'(s)\} = -tF(t)$

c) Find following inverse Laplace transformations:

(i) $\mathcal{L}^{-1}\left\{\dfrac{2s^3 + 10s^2 + 8s + 40}{s^2(s^2+9)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{s+2}{s^2(s+3)}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\ln\left(1 + \dfrac{1}{s^2}\right)\right\}$.

d) State the convolution theorem for the Laplace transformations. Apply convolution theorem to find the following inverse Laplace Transforations

(i) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}$.

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3. a) Let $\mathcal{L}\{Y(t)\} = y(s)$. Show, in the usual notation, that

(i) $\mathcal{L}\{Y'(t)\} = sy(s) - Y(0)$

(ii) $\mathcal{L}\{Y''(t)\} = s^2y(s) - sY(0) - Y'(0)$

b) Using the Laplace transform method solve the following system of ordinary differen equations:

$$\left.\begin{array}{l} \dfrac{dX}{dt} + \dfrac{dY}{dt} = t \\ \\ \dfrac{d^2X}{dt^2} - Y = e^{-t} \end{array}\right\}; \quad X(0) = 3,\ Y(0) = 0,\ X'(0) = -2.$$

Section B

1. a) Show,in the usual notation, that

(i) $\mathcal{L}\left\{\dfrac{\partial U(x,t)}{\partial t}\right\} = su(x,s) - U(x,0)$

(ii) $\mathcal{L}\left\{\dfrac{\partial^2 U(x,t)}{\partial t^2}\right\} = s^2u(x,s) - sU(x,0) - U_t(x,0)$ ·

(iii) $\mathcal{L}\left\{\dfrac{\partial U(x,t)}{\partial x}\right\} = \dfrac{du(x,s)}{dx}$ and

(iv) $\mathcal{L}\left\{\dfrac{\partial^2 U(x,t)}{\partial x^2}\right\} = \dfrac{d^2u(x,s)}{dx^2}$.

b) The faces $x = 0$ and $x = 1$ of a slab material for which thermal diffusivity $k = 1$ are kept at temperature 0 and 1 respectively until the temperature distribution becomes $u = x$. After time $t = 0$ both faces are held at temporature 0. Determine the temperature distribution at time t.

[You may assume that

$$\mathcal{L}^{-1}\left\{\frac{\sinh x\sqrt{s}}{s\sinh a\sqrt{s}}\right\} = \frac{x}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\frac{n^2\pi^2 t}{a}}\sin\left(\frac{n\pi x}{a}\right).]$$

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2. a) For $m > 0$, $n > 0$, the Beta function, $B(m,n)$ is defined as:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx.$$

(i) Using Laplace Transform methods show in the usual notation that
$B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

(ii) Use the above result to show that $2\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta = B(m,n)$.

b) Show folowings using the results in part (a)

(i) $\displaystyle\int_0^1 x^{3/2}(1-x)^2dx = \frac{\pi}{16}$ (ii) $\displaystyle\int_0^2 x^4\sqrt{4-x^2}dx = 2\pi$

(iii) $\displaystyle\int_0^{\pi/2} \sin^4\theta\cos^6\theta d\theta = \frac{3\pi}{512}$ (iv) $\displaystyle\int_0^{\pi/2}\frac{d\theta}{\sqrt{\tan\theta}} = \frac{\pi}{\sqrt{2}}$.

c) Use Laplace transform methods to show that,

$$\int_0^\infty \cos x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

[In the usual notation you may assume that $\Gamma(p)\Gamma(p-1) = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.]

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3. a) Suppose that $f(x)$ is a periodic function with the period $2L$ and its Fourier series is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ for $-L < x < L$; where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx$ for $n = 1, 2, 3, \ldots$. Show that f satisfies the Parsval's identity

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

b) Obtain the Fourier sine expansion of $f(x) = x(\pi - x)$, $0 \le x \le \pi$ in the form:

$$f(x) = \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right).$$

Hence, deduce that

(i) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}$,

(ii) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^6} = \frac{\pi^6}{960}$ and

(iii) $\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$.

Section C

1. a) The Legendre differential equation is given by $(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0$. Using the recurrence formula

$$(x^2-1)p_n'(x) = n\{xp_n(x) - p_{n-1}(x)\}.$$

Show in the usual notation that $xp_n'(x) - p_{n-1}'(x) = np_n(x)$; where $p_n(x)$ are the Legendre polynomials of order n.

Hence show that

$$p_{n+1}'(x) - p_{n-1}'(x) = (2n+1)p_n(x).$$

Deduce that

$$\int_x^1 p_n(x)dx = \frac{1}{(2n+1)}(p_{n-1}(x) - p_{n+1}(x)).$$

b) Show that the Rodrigue's formula

$$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

is a solution of the Legendre differential equation; where n is a positive integer.

2. a) The generating function of the Bessel function $J_n(x)$ is given by

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

By substituting $t = e^{i\phi}$, prove that the Bessel functions have the integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)d\phi.$$

Deduce that

(i) $J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}dt,$

(ii) $J_n'(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}.$

b) Show that

$$J_n(x) = \frac{1}{\sqrt{\pi}\Gamma(n+1/2)}\left(\frac{x}{2}\right)^n \int_0^{\pi} \cos(x\sin\phi)\cos^{2n}\phi d\phi$$

[You may assume that $J_{-n}(x) = (-1)^n J_n(x)$]

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3. The Hermit polynomial $H_n(x)$ of degree n associated with the Hermit differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

is defined by

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}$$

for all integral values of n and all real values of x.

Show that

(i) $H_n(x) = e^{x^2}(-1)^n \frac{d^n}{dx^n}(e^{-x^2})$

(ii) $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),\ n \geq 1$

(iii) $\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = \begin{cases} 0, & \text{if } m \neq n \\ \sqrt{\pi}2^n n!, & \text{if } m = n. \end{cases}$

Hence deduce that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}[H_n(x)]^2 dx = \sqrt{\pi}2^n n!\left(n + \frac{1}{2}\right)$$

Last Page