University of Ruhuna

Bachelor of Science General Degree
Level II (semester I) Examination – July 2015

Subject: Mathematics
Course Unit: MAT211β/MPM2113 (Linear Algebra)
Time: Two (02) hours

Answer <u>04 questions</u> only

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(1) (a) Let $A=[a_{ij}]$ be a non-singular n-square matrix. Prove, in the usual notation, that

(i) $A(adj(A)) = |A| I_n$ and

(ii) $|(adj(A))| = |A|^{n-1}$.

Using the result in part (i), deduce that $A^{-1} = \dfrac{(adj(A))}{|A|}$.

Verify (i) and (ii) above for the matrix

$$A=\begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

(b) Define an elementary matrix.
Prove that every non-singular square matrix A is expressible as a product of elementary matrices.

Find the inverse of the matrix

$$A=\begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 2 & 4 & -5 \end{pmatrix}$$

using elementary row operations.

(2) (a) Define normal form of a matrix A.

Reduced the matrix

$$A=\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 1 \end{pmatrix}.$$

to its normal form.

Hence find the rank of A.

(b) State the necessary and sufficient conditions that should be satisfied by a non-empty subset W of a vector space V to be a subspace of V.

Define a basis S for a vector space V.

Determine which of the following subsets is a subspace of $\Re^3$?

(i) $W = \{(x_1, x_2, 5) \mid x_1, x_2 \in \Re\}$
(ii) $W = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \Re\}$

For the case of W is a subspace find a basis for W and also find the dim(W).

(3) (a) A function T: $\mathcal{P}_5 \rightarrow \mathcal{P}_3$, where $\mathcal{P}_n$ is the linear space of polynomials of the nth order, is defined by

$$T(f(x)) = \frac{d^2}{dx^2}(f(x)).$$

Show that

(i) T is linear,
(ii) $\{1, x\}$ is a basis for Ker(T),
(iii) $\{1, x, x^2, x^3\}$ is a basis for Im(T),

Verify that $\dim(\mathcal{P}_5) = \dim(\ker(T)) + \dim(\mathrm{Im}(T))$.

(b) Explain what is meant by a diagonalizable matrix.

$$\text{Let } A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}.$$

Find a matrix P such that $P^{-1}AP$ is a diagonal matrix. Justify your answer.

(4) State clearly the condition under which a system of non-homogeneous linear equation

$$A_{m\times n} X_{n\times 1} = D_{m\times 1}$$

will have

(i) a unique solution
(ii) no solutions
(iii) Infinitely many solutions.

(a) Show that the system of linear equations

$$2x-2y+3z = a$$
$$3x+2y+z = b$$
$$4x+6y-z = c$$

has no solution unless $c = 2b - a$.

(b) Determine the values of α and β for which the system of linear equations

$$3x+y+2z = 3$$
$$2x-3y-z = -3$$
$$x+2y+\alpha z = \beta$$

has

(i) a unique solution
(ii) no solutions
(iii) Infinitely many solutions.

Find the solutions in both cases this linear system is consistent.

(5) Define what is meant by an Eigen value and corresponding Eigen vector of a matrix A_{nxn}.

Show that, if A and B are similar n×n matrices, then they have the same eigenvalues.

Consider the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

(i) Show that the characteristic equation of matrix A is given by

$$\lambda^3 - 12\lambda - 16 = 0.$$

(ii) Show that λ=4 is one Eigen value of the matrix.
(iii) Find the other Eigen values of the matrix.
(iv) Find the corresponding Eigen Vectors of the matrix.
(v) Find the basis for the each of the Eigen spaces.

(6) State the Cayley-Hamilton Theorem.

Explain what is meant by the minimal Polynomial of an $n \times n$ matrix A.

Let an 3×3 matrix be

$$. A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(i) Find the Characteristic equation of A.
(ii) Verify the Cayley-Hamilton Theorem for A.
(iii) Use the Cayley-Hamilton theorem to find the inverse of A
(iv) Find the characteristic roots of A.
(v) Hence find the minimal polynomial of A.
(vi) Use Cayley-Hamilton theorem to find A^5.
(vii) State whether the matrix A is derogatory or non-derogatory giving the reason.

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