University of Ruhuna

Bachelor of Science General Degree
(Level III) Semester I Examination - June/July 2015

Subject :Applied Mathematics
Course unit: AMT311β /MAM3113 (Numerical Analysis)

Time :Two (02) Hours

Answer four (04) Questions only
Only the calculators provided by the University are allowed to use.

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1. a) Define in the usual notation, the matrix norms, $\| A \|_1$, $\| A \|_2$ and $\| A \|_\infty$ of a matrix A of order n.

Find $\| A \|_1$ and $\| A \|_\infty$ for the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

b) Find A, B and C of the parabola $y = Ax^2 + Bx + C$ that passes through the points $(1,4), (2,7)$ and $(3,14)$ using Gauss Elimination method.

c) Apply Doolittle method and solve the system of linear equations given below.

$$\begin{pmatrix} 1 & 2 & -3 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 10 \\ 5 \end{pmatrix}.$$

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2. a) The system of linear equations $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, has an equivalent representation of the form $x = Tx + c$, where $T \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}^n$.
Suppose that the system has a unique solution $x^* \in \mathbb{R}^n$.
Consider the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ generated by the recurrence formula $x^{(k+1)} = Tx^{(k)} + c$, where $k = 0, 1, 2, \ldots$ with the initial approximation $x^{(0)}$.
Show that $x^{(n)} - x^* = T^n(x^{(0)} - x^*)$.

b) (i) Consider the system of linear equations $Ax = b$ and the decomposition $A = L + D + U$, where L, D and U represent the lower triangular, diagonal and upper triangular parts of A respectively.
Use this decomposition to formulate Gauss Seidel iteration method for solving $Ax = b$. Write down the general formula of the method.

Continued.

(ii) Consider the system of linear equations given by

$$\begin{pmatrix} 5 & -1 & 1 \\ 2 & 8 & -1 \\ -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \\ 3 \end{pmatrix}.$$

Taking the initial approximation as $x^{(0)} = (0\ 0\ 0)^T$, find the second iterate $x^{(2)}$ using the Gauss Seidel iteration method.

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3. a) Consider the rectangular region $D = \{(t, y) | 0 \leq t \leq 3, \ -1 \leq y \leq 1\}$.
Let $f(t, y) = t^2 y - 1$ with $y(0) = 1$.
Does f satisfy the Lipschitz condition on D ? If so find a Lipschitz constant.

b) Consider the initial value problem $y' = 2y - x; \quad y(0) = 1$.
Calculate the Picard Iterations $y_1(x)$, $y_2(x)$ and $y_3(x)$ for this initial value problem.

c) Consider the initial value problem $x''(t) + 4x'(t) + 5x(t) = 0$, with $x(0) = 3$ and $x'(0) = -5$.

(i) Transform this initial value problem into an equivalent system of first order differential equations.

(ii) Apply Euler's explicit method to the system with step size 0.1, and find the approximate values for x' and x at $t = 0.1, 0.2$ and 0.3 .

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4. a) Describe the Predictor-Corrector technique in approximating the solution of an initial value problem using modified Euler method and Euler's explicit method.
Using the Predictor-Corrector method described above to the initial value problem $y' = y - x^2$; $y(0) = 1$ with step size $h = 0.1$, obtain the approximate value of $y(0.1)$.

b) Consider the initial value problem $y' = f(x, y)$; $y(x_0) = y_0$ and the 2-stage Runge Kutta method in the usual notation,
$y_{n+1} = y_n + h(b_1 f(x_n, y_n) + b_2 f(x_n + c_2 h, y_n + a_{21} h f(x_n, y_n)))$; where h is the step size.

(i) Considering the Taylor expansion, obtain the equations $b_1 + b_2 = 1$, $b_2 c_2 = \frac{1}{2}$, $b_2 a_{21} = \frac{1}{2}$ for parameters b_1, b_2, c_2 and a_{21} so that the resulting Runge Kutta method is of order two.

(ii) Obtain the corresponding Runge Kutta scheme according to equations obtained in part (i) above, if $b_1 = \frac{1}{2}$.

(iii) Apply the scheme to the initial value problem $y' = e^x + xy^2$; $y(1) = 4$ with step size 0.1 and obtain the approximate value of $y(1.2)$.

Continued.

5. Let $u(x,t)$ be the solution of the heat equation, $u_{xx} = cu_t$ for $t > 0$ and $0 < x < a$ with boundary conditions $u(0,t) = T_0$ and $u(a,t) = T_1$, for $t > 0$ and, initial condition $u(x,0) = f(x)$, for $0 \leq x \leq a$.

Consider the discretization $x_i = ih$, for $i = 0, 1, 2, ..., n$ and $t_j = jk$, for $j = 0, 1, 2, ...$ where $h = \frac{a}{n}$, $k > 0$ are sufficiently small step sizes in x and t directions respectively.

Derive, in the usual notation, the explicit finite difference scheme,
$u_{i,j+1} = ru_{i+1,j} + (1 - 2r)u_{i,j} + ru_{i-1,j}$; where $r = \frac{k}{ch^2}$.

Draw the stencil for the scheme.

Consider the heat equation $u_{xx} = u_t$ for $t > 0$ and $0 < x < 1$ with boundary conditions $u(0,t) = 0, u(1,t) = 1$, for $t > 0$ and the initial condition $u(x,0) = x^3$, for $0 \leq x \leq 1$.

Using the above explicit finite difference scheme with $h = 0.2$ and $k = 0.02$, solve the heat equation $u_{xx} = u_t$ for $0 < t < 0.1$ and $0 < x < 1$.

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6. Let $u(x,t)$ be the solution of the wave equation, $u_{tt} = c^2 u_{xx}$ for $0 < x < a$ and $t > 0$ subject to the initial conditions $u(x,0) = f(x)$ and $\dfrac{\partial u}{\partial t}(x,0) = g(x)$, for $0 \leq x \leq a$ and boundary conditions $u(0,t) = \phi(t)$ and $u(a,t) = \phi(t)$, for $t \geq 0$.

Consider the discretization $x_i = ih$, for $i = 0, 1, 2, ..., n$ and $t_j = jk$, for $j = 0, 1, 2, ...$ where $h = \frac{a}{n}$, $k > 0$ are sufficiently small step sizes in x and t directions respectively.

Derive in the usual notation, the explicit finite difference scheme,
$u_{i,j+1} = -u_{i,j-1} + \alpha^2(u_{i+1,j} + u_{i-1,j}) + 2(1 - \alpha^2)u_{ij}$; where $\alpha = \frac{ck}{h}$.

Consider the wave equation, $u_{tt} = u_{xx}$ for $0 < x < 1$ and $0 < t < 1$ with the initial conditions $u(x,0) = x(1-x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$, for $0 \leq x \leq 1$ and boundary conditions $u(0,t) = u(1,t) = 0$, for $t \geq 0$.

Solve this wave equation using the explicit finite difference scheme derived above, with $h = k = 0.25$.
