

University of Ruhuna
Bachelor of Science General Degree Level III
(First Semester Examination)

June/July 2015

Subject: Applied Mathematics

Course Unit: MMA3113/AMT313β(Mathematical Methods in Physics and Engineering)

Time: Two (02) Hours

Answer 04 Questions only.

1. a) Define $\mathcal{L}\{F(t)\}$, the Laplace transform of a function $F(t)$, denoted by $f(s)$.

b) If $\mathcal{L}\{F(t)\} = f(s)$ then show that

(i) $\mathcal{L}\{e^{at}F(t)\} = f(s-a), s > a$ (ii) $\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}, s > |a|$

(iii) $\mathcal{L}\{tF(t)\} = -\frac{d}{ds}f(s)$ (iv) $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)du$

c) Find the following Laplace transformations:

(i) $\mathcal{L}\{(2e^{3t} \sin 4t)\}$, (ii) $\mathcal{L}\{t \sinh 4t\}$,
(iii) $\mathcal{L}\left\{\frac{\sinh t}{t}\right\}$, (iv) $\mathcal{L}\left\{\frac{e^{-2t} - e^{-3t}}{t}\right\}$,

(v) $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u} du\right\}$.

2. a) Define the inverse Laplace transform $\mathcal{L}^{-1}\{f(s)\}$ of $f(s)$.

b) Show, in the usual notation, that

$$\mathcal{L}^{-1}\{f'(s)\} = -tF(t)$$

c) Find following inverse Laplace transformations:

(i) $\mathcal{L}^{-1}\left\{\frac{2s^3 + 10s^2 + 8s + 40}{s^2(s^2 + 9)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2(s+3)}\right\}$

(iii) $\mathcal{L}^{-1}\left\{\ln\left(1 + \frac{1}{s^2}\right)\right\}$.

d) State the convolution theorem for the Laplace transformations. Apply convolution theorem to find the following inverse Laplace Transformations

(i) $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$.

3. a) Let $\mathcal{L}\{Y(t)\} = y(s)$. Show, in the usual notation, that
- $\mathcal{L}\{Y'(t)\} = sy(s) - Y(0)$
 - $\mathcal{L}\{Y''(t)\} = s^2y(s) - sY(0) - Y'(0)$
- b) Using the Laplace transform method solve the following system of ordinary differential equations:

$$\left. \begin{aligned} \frac{dX}{dt} + \frac{dY}{dt} &= t \\ \frac{d^2X}{dt^2} - Y &= e^{-t} \end{aligned} \right\}; \quad X(0) = 3, Y(0) = 0, X'(0) = -2.$$

4. a) Show, in the usual notation, that

- $\mathcal{L}\left\{\frac{\partial U(x,t)}{\partial t}\right\} = su(x,s) - U(x,0)$
- $\mathcal{L}\left\{\frac{\partial^2 U(x,t)}{\partial t^2}\right\} = s^2u(x,s) - sU(x,0) - U_t(x,0)$
- $\mathcal{L}\left\{\frac{\partial U(x,t)}{\partial x}\right\} = \frac{du(x,s)}{dx}$ and
- $\mathcal{L}\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} = \frac{d^2u(x,s)}{dx^2}$.

- b) The faces $x = 0$ and $x = 1$ of a slab material for which thermal diffusivity $k = 1$ are kept at temperature 0 and 1 respectively until the temperature distribution becomes $u = x$. After time $t = 0$ both faces are held at temperature 0. Determine the temperature distribution at time t .

[You may assume that

$$\mathcal{L}^{-1}\left\{\frac{\sinh x\sqrt{s}}{s \sinh a\sqrt{s}}\right\} = \frac{x}{a} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\frac{n^2\pi^2 t}{a^2}} \sin\left(\frac{n\pi x}{a}\right).]$$

5. a) For $m > 0, n > 0$, the Beta function, $B(m, n)$ is defined as:

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx.$$

- (i) Using Laplace Transform methods show in the usual notation that

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

- (ii) Use the above result to show that $2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = B(m, n)$.

b) Show followings using the results in part (a)

$$(i) \int_0^1 x^{3/2}(1-x)^2 dx = \frac{\pi}{16} \quad (ii) \int_0^2 x^4 \sqrt{4-x^2} dx = 2\pi$$

$$(iii) \int_0^{\pi/2} \sin^4 \theta \cos^6 \theta d\theta = \frac{3\pi}{512} \quad (iv) \int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan \theta}} = \frac{\pi}{\sqrt{2}}$$

c) Use Laplace transform methods to show that

$$\int_0^{\infty} \cos x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

[In the usual notation you may assume that $\Gamma(p)\Gamma(p-1) = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.]

6. a) Suppose that $f(x)$ is a periodic function with the period $2L$ and its Fourier series

is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$ for $-L < x < L$; where

$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$, $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ and $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$ for $n = 1, 2, 3, \dots$. Show that f satisfies the Parsval's identity

$$\frac{1}{L} \int_{-L}^L (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

b) Obtain the Fourier sine expansion of $f(x) = x(\pi - x)$, $0 \leq x \leq \pi$ in the form:

$$f(x) = \frac{8}{\pi} \left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots \right).$$

Hence, deduce that

$$(i) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32},$$

$$(ii) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^6} = \frac{\pi^6}{960} \text{ and}$$

$$(iii) \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$