UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 5 Examination in Engineering: August 2024

Module Number: IS5306 **Module Name: Numerical Methods**

[Three hours]

[Answer **all questions,** each question carries 12 marks]

Q1. a) Write down the bisection theorem.

[2 Marks]

b) i.) Find the interval in which the smallest positive root of the following equation lies

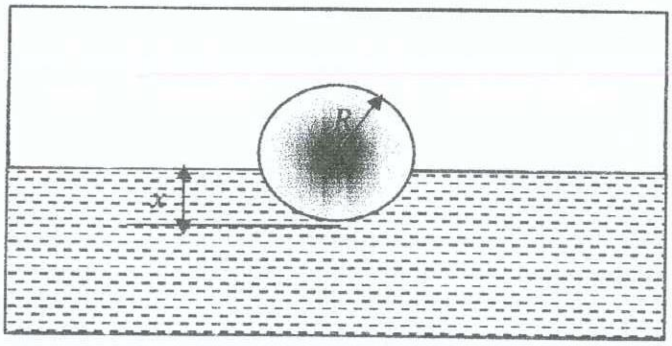
$$f(x) = x^3 + 4x^2 - 10 = 0.$$

ii.) Determine the roots using the bisection method. Consider the tolerance as 0.001.

[4 Marks]

c) Assume that you are employed at 'AquaCraft,' a company specializing in water-based products. One of your tasks involves designing floats for toilet systems produced by 'Hygiene Innovations.' These floats, similar to those used in standard toilet tanks, have a floating ball with specific characteristics. The ball you're working with has a specific gravity of 0.6 and a radius (R) of 5.5 cm. Your assignment is to determine how deeply the ball will submerge when placed in a container filled with water. The equation that gives the depth x in meters to which the ball is submerged under water is given by

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$



Use the Newton-Raphson method of finding roots of equations to find

i.) the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation. Assume that the intial guess of the root of $f(x)$ is 0.05 m.

ii.) the absolute relative approximate error at the end of each iteration.

[6 Marks]

Q2. a) The Lagrangian interpolating polynomial of degree n that passes through $n+1$ data points (x_0,y_0), (x_1,y_1),…………, (x_n,y_n) is defined as $P_n(x)=\sum_{i=0}^{n} y_i L_i(x)$. Where

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j},$$

Show that

$$L_i(x_j) = 1 \qquad \text{when } j = i \text{ and}$$
$$L_i(x_j) = 0 \qquad \text{when } j \neq i$$

[2 Marks]

b) Write down the third derivative approximation equation for the

i.) Newtons Forward difference.
ii.) Newtons Backward difference.
iii.) Newtons Divided difference.

[3 Marks]

c) Find unique polynomial $P(x)$ of degree 2 such that

$$P(1) = 1, \qquad P(3) = 27, \qquad P(4) = 64$$

using each of the following methods

i.) Lagrange Interpolation formula
ii.) Newtons Divided difference formula

Evaluate $P(1.5)$.

[3 Marks]

d) Solve the system of linear quations

$$\begin{aligned} 2x - y &= 1 \\ -x + 2y - z &= 0 \\ -y + 2z - w &= 0 \\ -z + 2w &= 1 \end{aligned}$$

i.) Using Jacobi method.
ii.) Gauss-Seidel Iteration method.

Iterate three times with $x^{(0)} = [0.5 \quad 0.5 \quad 0.5 \quad 0.5]^T$

[4 Marks]

Q3. a) In a grocery store, a brand of cereal advertises that every box contains at least 250 grams of cereal. The probability that a randomly selected box of this cereal contains 250 grams or more is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515\, e^{-0.3881(y-252.2)^2}\, dy$$

approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515\, e^{-0.3881(y-252.2)^2}\, dy$$

i.) Use the Trapezoidal rule to find the probability that there are 250 grams or more cerial. Assume that there are two sub intervals. Approximate your answer to the five decimal places.

ii.) Find the absolute relative error for part (i) in above. Assume that the true value of the above integral as 0.97377.

[6 Marks]

b) If an oscillator needs to operate within 5% of its target frequency of 1 kHz, the likelihood of this occurring can be evaluated by calculating the total area under the normal distribution curve for the specified frequency range is given below.

$$(1-\alpha)=\int_{-2.15}^{2.9}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

i.) Evaluate the integral using three-point Gauss quadrature rule to find the frequency.

ii.) Find the absolute relative error for part (i) in above . Assume that the exact value of the above integral is 0.98236.

Refer the following table for the weighting factors and function argument values.

Point	*Weight Factors*	*Function Arguments*
2	$C_1 = 1.0000$	$t_1 = -0.5773$
	$C_2 = 1.0000$	$t_2 = 0.5773$
3	$C_1 = 0.5555$	$t_1 = -0.7746$
	$C_2 = 0.8888$	$t_2 = 0.0000$
	$C_3 = 0.5555$	$t_3 = 0.7746$

[6 Marks]

Q4. a) Briefly explain the following by giving an example for each item.

i.) Ordinary differential equations (ODE)
ii.) Boundary value problems (BVP)

[2 Marks]

b) A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by,

$$\frac{d\theta}{dt}=-2.2067\times10^{-12}(\theta^4-81\times10^8)$$

Find the temperature at $t = 8$ minutes, using Runge-Kutta 4th order method. Assume a step size of $h = 4$ minutes.

[7 Marks]

c) Using the following ordinary differential equation with given initial conditions and step size of h, show that the error of 4th order Runge-Kutta method is of order 5 [$O(h^5)$].

$$\frac{dy}{dx} = y, \qquad y(0) = 1$$

[3 Marks]

Q5. a) Classify the following partial differential equations as hyperbolic, parabolic, or elliptic.

i.) $8\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0$

ii.) $\alpha\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

iii.) $\frac{\partial^2 u}{\partial x^2} + x^2\frac{\partial^2 u}{\partial y^2} = 0, \quad x \neq 0$

iv.) $2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial t} + 3\frac{\partial^2 f}{\partial t^2} + 4\frac{\partial f}{\partial x} + \cos(2t) = 0$

[3 Marks]

b) Use Crank-Nicolson method to solve the partial differential equation,

$$\frac{\partial T(x,t)}{\partial t} = 0.4\frac{\partial^2 T(x,t)}{\partial x^2}$$

for $0 \leq x \leq 1$ and $0 \leq t \leq 1$,

with the initial conditions,

$T(x, 0) = 100x$ for $0 \leq x \leq 0.4$; $T(x, 0) = 50(1.5 - x)$ for $0.4 < x \leq 1$

and the boundary conditions,

$T(0,t) = 0$
$T(1,t) = 25$.

Use, $h = 0.2$ and $k = 0.5$, where h and k are step sizes along x and t axes respectively.

[6 Marks]

c) Briefly explain the procedure of finite difference solution technique for solving Laplace equation $\left(\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0\right)$ highlighting any differences with the solution technique used in part (b).

[3 Marks]