University of Ruhuna

Bachelor of Science Special Degree Level I (Semester I) Examination - July 2016

Subject: Statistics
Course Unit: MSP3193 (Bayesian Inference and Decision Theory)

Time :Three (03) Hours

Answer 04 Questions only.

---

1. Let X, Y, Z have joint pdf $f(x, y, z) = 2(x + y + z)/3,\ \ 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1$, zero otherwise.

   (a) Find the marginal probability density function of $X, Y,$ and Z.

   (b) Compute $Pr(0 < X < \frac{1}{2},\ 0 < Y < \frac{1}{2},\ 0 < Z < \frac{1}{2})$ and $Pr(0 < X < \frac{1}{2})$ $Pr(0 < Y < \frac{1}{2})$ and $Pr(0 < Z < \frac{1}{2})$.

   (c) Are $X, Y,$ and Z are independent? (You may use part (b))

   (d) Find the conditional distribution of X and Y, given $Z = z$, and compute $E(X + Y/z)$.

---

2. Let y be the number of heads in n spins of a coin, whose probability of heads is θ.

   (a) If your prior distribution for θ is uniform on the range $[0, 1]$, derive your predictive distribution for y,

   $$Pr(y = k) = \int_0^1 Pr(y = k|\theta)\ d\theta,$$

   for each $k = 0, 1, \ldots, n$.

   (b) Suppose you assign a $Beta(\alpha, \beta)$ prior distribution for θ, and then you observe y heads out of n spins. Show algebraically that your posterior mean of θ always lies between your prior mean, $\frac{\alpha}{\alpha+\beta}$ and the observed relative frequency of heads $\frac{y}{n}$.

(c) Show that, if the prior distrbution on θ is uniform, the posterior vraiance of θ always less than the prior vaiance.

(d) Given an example of a $Beta(\alpha, \beta)$ prior distribution and data y, n, in which the posterior variance of θ is higher than the prior variance.

---

3. A random sample of n students is drawn from a large population, and their weights are measured. The average weight of the n sampled students is $\bar{y} = 150$ pounds. Assume the weights in the population are normally distributed with unknown mean θ and known standard deviation 20 pounds. Suppose your prior distribution for θ is normal with mean 180 and standard deviation 40.

(a) Give your posterior distrbution for θ. (Your answer will be a function of n).

(b) A new student is sampled at random from the same population and has a weight of $\tilde{y}$ pounds. Give a posterior predictive distribution for $\tilde{y}$. (Your answer will be a function of n).

(c) For $n = 10$, give a 95% posterior interval for θ and a 95% posterior predictive interval for $\tilde{y}$.

(d) Do the same for $n = 100$.

---

4. Consider the model

$$
\begin{aligned}
X_i|\theta &\sim \text{iid } N(\theta, \sigma^2), \text{ where } \sigma^2 \text{ is known} \\
\theta &\sim N(\theta_0, \sigma_0^2) \text{ where } \theta_0 \text{ and } \sigma_0^2 \text{ are known}
\end{aligned}
$$

(a) Find the sufficient statistic for θ.

(b) Write down an equivalent formulation of the model in terms of the suuficient statistic and θ.

(c) Find the posterior distribution of θ.

(d) Show that the posterior mean and posterior variance are given by

Continued.

$$(\frac{\sigma_0^2}{\sigma_0^2 + \sigma^2/n})\,\bar{x} + (\frac{\sigma^2/n}{\sigma_0^2 + \sigma^2/n})\,\theta_0 \text{ and } \frac{(\sigma^2/n)\,\sigma_0^2}{(\sigma_0^2 + \sigma^2/n)}$$

respectively.

---

5. (a) What is meant by a conjugate family of distributions?

(b) Consider the model

$$X_i|\theta \sim \text{iid Poisson}(\theta)$$
$$\theta \sim \text{Gamma}(\alpha, \beta), \text{ where } \alpha \text{ and } \beta \text{ are known.}$$

(i) Find the posterior distribution of θ.

(ii) Is this conjugate family?

(c) Find the prior predictive distribution for a single observation, y, for the model in part (b). Identify the form of this distribution?

(d) In many applications, it is convenient to extend the Poisson model for data points $x_1, x_2, \ldots, x_n$ to the form

$$x_i \sim \text{Poisson}(y_i\theta)$$

where the values y_i are known positive values of an explanatory variable y.

(i) Give an example for θ and y_i for such a model.

(ii) Using $\theta \sim \text{Gamma}(\alpha, \beta)$ in part (b), write down the posterior distribution of θ. (You do not need to derive this.)

---

6. Suppose a random sample is taken from an exponential distribution with mean λ. If we assign the usual noninformative prior $g(\lambda) \propto 1/\lambda$, then the poeterior density is given, up to a proportionality constant, by

$$g(\lambda/\text{data}) \propto \lambda^{-n-1} \exp\{-s/\lambda\},$$

where n is the sample size and s is the sum of the observations.

(a) Show that if we transform λ to $\theta = 1/\lambda$, then θ has a gamma density with shape parameter n and rate parameter s.

(b) In a life-testing illustration, five bulbs are tested with observed burn times (in hours) of
$751, 594, 1213, 1126,$ and 819. Using the R function *rgamma* write down a short R codes to simulate 1000 values from the posterior distribution of θ.

(c) Write down a short R codes to transform these simulated draws in part (b) and to obtain a simulated sample from the posterior distribution of λ.

(d) Explian briefly how you compute the posterior probability that λ exceeds 1000 hours.
