University of Ruhuna

Bachelor of Science General Degree
Level III (Semester I) Examination - July 2016

Subject: Mathematics
Course Unit: MAT 311β/MSP 311β(Group Theory)

Time :Two (02) Hours

Answer 04 Questions only.

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1. a) Show that the set of all non zero complex numbers forms a group under multiplication defined by,
$(a+bi)(c+id) = (ac-bd) + i(ad+bc)$.

b) Consider the set,
$G = \{(a,b)|a,b$ rationals, $a \neq 0\}$, and define $*$ on G by
$(a,b)*(c,d) = (ac, ad+b)$. Show that G forms a group under the operation $*$.
Does G form an abelian group? Justify your answer.

c) (i) Show that the identity element of a group G is unique.

(ii) Let H be a subgroup of a group G. Then prove that the identity of H is the same as the identity of G.

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2. a) Let G be a group and let H and K be subgroups of G. Prove that
$HK = \{hk \in G | h \in H, k \in K\}$ is a subgroup of G if and only if $HK = KH$.

b) The map $f : \mathbb{R} \to \mathbb{R}$ is defined by $f_{ab} = ax + b$, where $a, b, \in \mathbb{R}, a \neq 0$.
Let $G = \{f_{ab} : a, b \in \mathbb{R}, a \neq 0\}$ be a group under composition of mappings.

(i) Find the identity of f_{ab} in G and the inverse of f_{ab} in G.

(ii) Does $N = \{f_{1b} : b \in \mathbb{R}\}$ form a normal subgroup of G? Justify your answer.

c) Express each of the following permutations as a single cycle or as the product of disjoint cycles.
Hence find the order of each

(i) (234)(135)(25)

(ii) (125)(236)

(iii) (142)(235)(134)

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3. a) Prove that the order of a finite cyclic group is equal to the order of its generator.

b) (i) Find the order of each of the elements 2,3,4 in the group $(\mathbb{Z}_8, \oplus_8)$.

(ii) Find all the generators of the group $(\mathbb{Z}_{10}, \oplus_{10})$.
(You may assume that if a cyclic group generated by an element a of order n, then a^m is a generator of G if and only if $(m, n) = 1$.)

c) Find whether the following permutations are even or odd.

(i) (312)

(ii) (53241)

(iii) (2134)

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4. a) Prove that any two right(left) cosets of a subgroup H in a group G are either disjoint or identical.

b) Let $(\mathbb{Z}_6, \oplus_6)$ be a group. Find the distinct left cosets of the subgroup $H = \{0, 2, 4\}$ of G.
Verify the result that G is equal to the union of all left cosets.

c) Let a and b be arbitrary distinct elements of a group G and, H be any subgroup of G. Show that
$Ha = Hb \Leftrightarrow ab^{-1} \in H$.

d) Show that a subgroup of index 2 in a group G is a normal subgroup of G.

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5. a) Let G and G' be two groups and $f : G \to G'$ be a homomorphism.
Define the kernal of f(Ker f).
Prove that,

(i) Ker f is a normal subgroup of G;

(ii) f is one-one if and only if Ker f $= \{e\}$, where e is the identity of G.

b) (i) Let $(G, *)$ and $(G', \circ)$ be two groups and $G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; ad - bc \neq 0, a, b, c, d, \in \mathbb{R}\}$ be a group under matrix multiplication. Let G' be the group of non-zero real numbers under multiplication.
Show that the map $\phi : G \to G'$ defined by $\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ is homomorphism.

(ii) Consider the mapping $\mathbb{C}^* \to \mathbb{C}^*$ given by $f(x) = x^4$, where $\mathbb{C}^*$ is the set of non-zero complex numbers forms a group under multiplication.
Show that f is a homomorphism.
Find Ker f.
What is the kernal of f, if f is defined by $f(x) = x^n$?

Continued.

6. a) Let G be a group and let G' be the commutator subgroup of G generated by elements of the form $x^{-1}y^{-1}xy$, where x and y are any two elements of G. Prove that

(i) G is abelian $\Leftrightarrow G' = \{e\}$;

(ii) G' is normal in G;

(iii) $G|G'$ is abelian.

b) Let G, G' be groups and $f : G \to G'$ be an onto homomorphism with $K = Kerf$. Show that $G|K \cong G'$
