University of Ruhuna

Bachelor of Science Special Degree
Level II (Semester II) Examination

December 2016

Subject: Mathematics
Course Unit: MSP4224 (Introduction to Stochastic Analysis with Applications)
Time: Three(03) Hours

Answer **ALL Questions.** Calculators will be provided.

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1. a) Consider the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$ be two events. By taking all possible unions, intersections and complements, write down the σ-field $\mathcal{F}$ generated by A and B.

b) Consider the sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Define a probability measure on $(\Omega, 2^\Omega)$ by

$$P(\omega_1) = \frac{1}{3},\ P(\omega_2) = \frac{1}{6},\ P(\omega_3) = \frac{1}{3},\ P(\omega_4) = \frac{1}{6},$$

and a random variable X by

$$X(\omega) = I_{\{\omega_1,\omega_2\}}(\omega) = \begin{cases} 1 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{otherwise.} \end{cases}$$

Now define the σ-fields $\mathcal{F} = \sigma\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and $\mathcal{G} = \sigma\{\{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}\}$.

(i) Compute the expectation of X, i.e. $E[X]$ and the conditional expectation $E[X|\mathcal{F}]$.

(ii) Compute $E[X|\mathcal{G}]$. Is X independent of $\mathcal{G}$?

c) Define the one dimensional standard Brownian motion $(B_t,\ t \geq 0)$.
Consider the stochastic process $X_t = B_t^2 - t,\ t \geq 0$. We associate with X_t an increasing stream of information about the structure of the process which is represented by the σ-fields $\mathcal{F}_s = \sigma\{B_x,\ x \leq s\}$.
By considering the two cases $s \geq t$ and $s < t$ separately and stating clearly any results you may use, show that

$$E[X_t|\mathcal{F}_s] = X_{\min(s,t)}.$$

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2. a) A Poisson process with intensity $\lambda = 1$ is a stochastic process $\{N_t,\ t \geq 0\}$, with initial value $N_0 = 0$ satisfying

- if $0 = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n$, then $N_{t_k} - N_{t_{k-1}}$, $1 \leq k \leq n$ are independent. This implies that if $\mathcal{F}_t = \sigma\{N_t,\ t \geq 0\}$ is the filtration generated by N, then $N_t - N_s$ is independent of $\mathcal{F}_s$.
- $N_t - N_s$ is Poisson distributed with parameter $t - s$. The parameter is same as the expected value and variance.

Define the process M by $M_t = N_t - t$. This process is then a martingale with respect to the filtration generated by N. Now consider the process $M_t^2 - t$. Show that

(i) $E[|M_t^2 - t|] < \infty$.

(ii) the process $M_t^2 - t$ is a martingale with respect to the filtration generated by N, i.e. $E[M_t^2 - t|\mathcal{F}_s] = M_s^2 - s$.

b) State the isometry property for the Ito stochastic integral.
Consider the processes defined by

$$X_t = \int_0^t \cos(u)\, dB_u \text{ and } Y_t = \int_0^t \sin(u)\, dB_u,$$

where B denotes one dimensional Brownian motion.

(i) Prove the identity $E[X_t^2 + Y_t^2] = t$, where E denotes expectation.

(ii) Use the simple Ito formula to prove that $M_t = X_t^2 + Y_t^2 - t$ is a martingale, i.e. dM_t does not have a dt term.

(iii) Find all the values of t such that the covariance of the processes X_t and Y_t is zero, i.e. $\text{Cov}(X_t, Y_t) = 0$.

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3. Define a linear Ito stochastic differential equation.

Apply the simple Ito formula to show that the linear Ito stochastic differential equation

$$X_t = X_0 + (\sigma^2 - \lambda)\int_0^t X_s\, ds - \sigma \int_0^t X_s\, dB_s,\ t \in [0, T],$$

where λ and σ are positive constants and $(B_t,\, t \geq 0)$ is the one dimensional standard Brownian motion, has a solution of the form

$$X_t = X_0 \exp\{(\frac{1}{2}\sigma^2 - \lambda)t - \sigma B_t\}.$$

(Hint: You may apply the method of separation of variables appropriately.)

Consider the stochastic differential equation (SDE)

$$dZ_t = (\lambda Z_t - Z_t^2)\, dt + \sigma Z_t\, dB_t,\ Z_0 = z_0,$$

where λ and σ are positive constants and $(B_t,\, t \geq 0)$ is the standard Brownian motion.

(i) Use $Y_t = \dfrac{1}{Z_t}$ in the simple Ito formula to show that the above SDE can be written in the linear form

$$dY_t = (1 + (\sigma^2 - \lambda)Y_t)\, dt - \sigma Y_t\, dB_t.$$

(ii) Suppose the linear SDE in part (i) has a solution of the product form $Y_t = U_t V_t$, where

$$dU_t = (\sigma^2 - \lambda)U_t\, dt - \sigma U_t\, dB_t,\ U_0 = 1,$$

and

$$dV_t = a_t\, dt + b_t\, dB_t,\ V_0 = Y_0.$$

Here a_t and b_t are adapted processes which can be found by coefficient matching.

(α) Use the Ito product formula to find dY_t.

(β) Identify a_t and b_t.

(γ) Obtain U_t and V_t.

(iii) Show that

$$Z_t = \frac{z_0 \exp\{(\lambda - \frac{1}{2}\sigma^2)t + \sigma B_t\}}{1 + z_0 \int_0^t \exp\{(\lambda - \frac{1}{2}\sigma^2)s + \sigma B_s\}\, ds}.$$

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4. a) (i) Let X_t have stochastic differential $dX_t = B_t\, dt + t\, dB_t$, $X_0 = 0$. It is obvious that $X_t = tB_t$ satisfies this equation.
Let Y_t have stochastic differential

$$dY_t = \frac{1}{2} Y_t\, dt + Y_t\, dB_t,\ Y_0 = 1.$$

Using the simple Ito formula, show that $Y_t = e^{B_t}$ satisfies this equation.
Now use the Ito product formula to find $d(X_t Y_t)$.

(ii) Compute the stochastic differential of

$$\sin(B_t^2) \int_0^t e^s\, dB_s$$

appropriately using the simple Ito formula and Ito product formula. Here, B is one dimensional standard Brownian motion.

b) Let S be the price of a particular asset at time t. After a (short) time interval dt, the asset price changes by dS, to $S + dS$. Rather than measuring the absolute change dS, we measure the *return* on the asset which is defined to be $\frac{dS}{S}$ and is given by

$$\frac{dS}{S} = \mu\, dt + \sigma\, dX,$$

where μ and σ are called *drift* and *volatility*, respectively. The quantity dX is a random variable having a normal distribution with mean zero and variance dt.

(i) Assuming $\sigma = 0$, show that the asset price is given by $S = S_0 e^{\mu t}$, where S_0 is the asset price at time $t = 0$.

(ii) The price S of a particular share today is LKR 100. Construct a time series for the share price (i.e. find S_1, S_2, S_3, S_4, S_5 taking dX as -0.05, 0.12, 0.08, 0.16, 0.06, respectively) over five intervals if $\mu = 0.8, \sigma = 0.6, dt = \frac{1}{300}$.
Also demonstrate the time series graphically.
(Hint: You may use values of S_1, S_2, S_3, S_4 and S_5 in the graph approximately correct to one decimal place.)
