University of Ruhuna

Bachelor of Science General Degree

Level III (Semester II) Examination - November-2016

Subject: Mathematics

Course Unit: MAT322β (Complex variables)

MSP323β

Time :Two (02) Hours

Answer <u>04 Questions</u> only.

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1. (a) The sets A and B are defined as follows:

$$A = \{z = x + iy \in \mathbb{C}; x, y \in \mathbb{R} \text{ and } x > \alpha\},$$

and

$$B = \{z = x + iy \in \mathbb{C}; x, y \in \mathbb{R} \text{ and } \alpha < x < \beta\},$$

where $\alpha, \beta \in \mathbb{R}$ and $0 < \alpha < \beta$.

(i) Sketch the sets A and B in the complex plane.

(ii) Explain whether each of the above sets is (a) an open set, (b) a connected set, (c) a domain.

(b) Sketch the regions given by

$$(i)\ |z - i + 2| = 5, \qquad (ii)\ |z - i + 2| > 5,$$
$$(iii)\ |z + 2i| \leq 1, \qquad (iv) Im\ z \geq 0.$$

in the complex plane

(c) Let a be a complex number. Find $\lim_{n\to\infty} a^n$ when $|a| < 1$. Does $\lim_{n\to\infty} a^n$ exist, when $|a| > 1$?. Justify your answer.

(d) By substituting $z = re^{i\theta}$, examine the continuity of the function

$$f(z) = \begin{cases} \frac{Re z^2}{|z|^2} & z \neq 0 \\ 0 & z = 0, \end{cases}$$

at $z = 0$.

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2. (a) In the usual notation, obtain Cauchy-Riemann equations in polar form

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}, \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r},$$

for the differentiable function $f(z) = u(r, \theta) + iv(r, \theta)$.

Hence, show that the function $f(z) = z^n$, where n is any integer satisfies the Cauchy-Riemann equations.

(b) Find the complex conjugate harmonic function $v(x, y)$ of $u(x, y) = x + y^3 - 3x^2y$ and the corresponding analytical function $f(z)$.

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3. (a) Evaluate each of the integrals:

(i) $I_1 = \oint_{C_1} z^n dz$, $n = 0 \pm 1, \pm 2 \ldots$ where $C_1 : |z| = r$ is traversed in the counter-clockwise direction,

(ii) $I_2 = \oint_{C_2} (z - z_0)^n dz$, $n = 0 \pm 1, \pm 2 \ldots$ where $C_2 : |z - z_0| = r$ is traversed in the counter-clockwise direction,

(iii) $I_3 = \oint_{C_3} \frac{9}{z(z-3)} dz$, where C_3 is the curve $|z - 3| = 4$.

(iv) $I_3 = \oint_{C_4} \frac{9}{z(z-3)} dz$, where C_4 is the curve $|z - 3| = 2$.

(b) State the Cauchy integral theorem for the integration of complex function. Using theorem and extension evaluate the integral $I = \oint_{|z|=1} f(z)dz$, where

$$f(z) = \frac{3z + 4}{z(z + 2)}.$$

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4. (a) State the Cauchy integral formula. Using formula, evaluate each of the following integrals:

(i) $\oint_{|z|=2} \frac{z^n}{z-1} dz, \quad n \geq 0,$

(ii) $\oint_C \frac{z+1}{z^2-9} dz$, for the cases $(\alpha)\ C : |z - 3| = 1$, $(\beta)\ C : |z + 3| = 1$, $(\gamma)\ C : |z| = 4$.

(b) State the Cauchy integral formula for derivatives. Evaluate each of the following integrals:

(i) $\oint_{|z|=1} \frac{e^z}{z^m} dz, \quad m \in (-\infty, 0] \cup [1, \infty),$

(ii) $\oint_C \frac{dz}{z(z^2-4)e^z} dz$ where $C : |z - 1| = 2$.

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5. (a) (i) By using Cauchy integral formula, in the usual notation, obtain the Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ where } a_n = \frac{1}{n!} f^{(n)}(z_0),$$

for an analytic function $f(z)$ inside a circle $|z - z_0| = R(> 0)$.

(ii) Find the Taylor series expansion and its radius of convergence of the function

$$f(z) = \frac{1}{(z + 3i)(z + 1)},$$

about $z = 0$.

(b) Find all the possible Taylor and Laurent series expansions about $z = 0$ of the function

$$f(z) = \frac{1}{(z^2 - 1)(z^2 - 4)}.$$

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6. (a) State clearly the Cauchy residue theorem.
Suppose complex function is given by

$$f(z) = \frac{e^z - 1}{z(z-1)(z-i)^3},$$

(i) find all the singular points of f,

(ii) find the residues each of the singular points,

(iii) use the Cauchy Riemann theorem to evaluate the integral $\oint_C f(z)dz$ where $C : |z| = 2$.

(b) By Cauchy Riemann theorem, Evaluate the integral

$$\int_0^{\infty} \frac{\sin 2x}{x(x^2 + 3^2)} dx.$$
