University of Ruhuna - Faculty of Science

Bachelor of Science General Degree - Level I (Semester I) Examination - September 2017

Subject: Mathematics
Course Unit: MAT111β / MAM1113 (Vector Analysis) **Time: Two (02) Hours**

Answer Four (04) questions only

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1. Let P be any point on a straight line ℓ in the three-dimensional space with position vector $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ relative to the origin O. Consider any two fixed points A and B on this straight line with position vectors $\underline{a}$ and $\underline{b}$ respectively relative to the origin O.

Show that the vector equation of the straight line ℓ can be written as

$$\underline{r} = \underline{a} + \lambda \underline{c},$$

where λ is a parameter and $\underline{c} = \underline{b} - \underline{a}$.

Write down the coordinates (x, y, z) of any point on the straight line ℓ if $\underline{a} = (a_1, a_2, a_3)$ and $\underline{c} = (c_1, c_2, c_3)$.

Consider the following two straight lines ℓ_1 and ℓ_2 in the three-dimensional space:

ℓ_1 : passes through the point $(0, 2, -1)$ with direction vector $(1, 1, 2)$
ℓ_2 : passes through the point $(1, 0, -1)$ with direction vector $(1, 1, 3)$

(a) Write down the vector equations of ℓ_1 and ℓ_2 choosing α and β respectively as parameters. Hence obtain the parametric Cartesian equations of ℓ_1 and ℓ_2.

(b) Show that ℓ_1 and ℓ_2 do not intersect each other.

(c) Let P and Q be the two points on ℓ_1 and ℓ_2 respectively such that PQ is the shortest distance between ℓ_1 and ℓ_2. Find $\overrightarrow{PQ}$ and show that $|\overrightarrow{PQ}| = \dfrac{3\sqrt{2}}{2}$ units.

(d) Find the coordinates of the points P and Q.

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2. Define, in the usual notation,

(i) gradϕ of a scalar field $\phi(x, y, z)$;

(ii) div$\underline{F}$ and curl$\underline{F}$ of a vector field $\underline{F}(x, y, z) = F_1\,\underline{i} + F_2\,\underline{j} + F_3\,\underline{k}$.

(a) The temperature T at any point in the three-dimensional space is given by $T = xy + yz + zx$. Find

(i) the directional derivative of T at the point (1,2,3);

(ii) the component of the directional derivative of part (i) in the direction of the vector (-3,2,4).

(b) If a vector field $\underline{F}$ is given by $\underline{F} = (3x^2y - z)\,\underline{i} + (xz^3 + y^4)\,\underline{j} - 2x^3z^2\,\underline{k}$, find grad div $\underline{F}$ at (2,-1,0).

(c) The motion of a fluid is described by the velocity field

$$\underline{v} = (y\sin z - \sin x)\,\underline{i} + (x\sin z + 2yz)\,\underline{j} + (xy\cos z + y^2)\,\underline{k}.$$

(i) Show that $\underline{v}$ is irrotational.

(ii) Show that the velocity potential $\phi(x,y,z)$ is given by

$$\phi(x,y,z) = xy\sin z + \cos x + y^2 z + C,$$

where C is a constant.

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3. Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = |\underline{r}|$ be the distance to the point $P \equiv (x,y,z)$ from the origin O. Furthermore, all the notations have usual meanings.

(i) Find $\operatorname{grad} \frac{1}{r}$.

(ii) Compute

$$\operatorname{curl}\left(\underline{k} \times \operatorname{grad}\frac{1}{r}\right)$$

and

$$\operatorname{grad}\left(\underline{k} \cdot \operatorname{grad}\frac{1}{r}\right),$$

where $\underline{k}$ is the unit vector in the direction Oz.

(iii) Hence, show that

$$\operatorname{curl}\left(\underline{k} \times \operatorname{grad}\frac{1}{r}\right) + \operatorname{grad}\left(\underline{k} \cdot \operatorname{grad}\frac{1}{r}\right) = \underline{0}.$$

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4. (a) Consider the vector field $\underline{F} = (e^x z - 2xy)\,\underline{i} + (1 - x^2)\,\underline{j} + (e^x + z)\,\underline{k}$.

(i) Show that the line integral $\int_C \underline{F} \cdot d\underline{r}$ is independent of path C. Here $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$.

(ii) Find a scalar function f such that $\underline{F} = \operatorname{grad} f$.

(iii) Hence evaluate the line integral $\int_C \underline{F} \cdot d\underline{r}$ along a curve from (0,1,-1) to (2,3,0).

(b) State the Green's theorem in the plane.

Verify Green's theorem for

$$\int_C (xy + y^2)\,dx + x^2\,dy,$$

where C is the closed curve of the region bounded by $y = x$ and $y = x^4$.

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5. State the divergence theorem of Gauss.

It is given that the vector field $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and S is the surface of the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$. Verify the divergence theorem of Gauss.

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6. State the Stokes' theorem.

Verify the Stokes' theorem for the vector field $\underline{F} = z\underline{i} + (2x + z)\,\underline{j} + x\underline{k}$ taken over the triangular surface S in the plane $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$ bounded by the planes $x = 0, y = 0$ and $z = 0$. Take the boundary of the above triangular surface as the path of the line integral.
