University of Ruhuna

Bachelor of Science Special Degree

Level I (Semester I) Examination - August 2017

Subject: Mathematics

Course Unit: MSP312β (Real Analysis-III) **Time:** Two (02) Hours

Answer **ALL Questions.** Calculators will be provided.

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1. a) Define the Euclidean norm $||\underline{x}||$ of a vector $\underline{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Let $0 < \epsilon \in \mathbb{R}$ and S be a subset of $\mathbb{R}^n$. Define the following:

   (i) an open ball $B_\epsilon(\underline{x})$ with center $\underline{x}$ and radius ϵ,

   (ii) an interior point $\underline{x}$ in S.

   Explain what you mean by "a subset $S \in \mathbb{R}^n$ is open?

   Determine whether the set $\{(x, 0) \in \mathbb{R}^2 | x \in \mathbb{R}\} \subset \mathbb{R}^2$ is open or not.

   b) Given any two vectors $\underline{x}, \underline{y} \in \mathbb{R}^n$, state and prove the Cauchy-Schwarz inequality. Write down the above inequality using the components of $\underline{x}$ and $\underline{y}$ when $\underline{x} = (x_1, x_2, \ldots, x_n)$ and $\underline{y} = (y_1, y_2, \ldots, y_n)$.

   (i) Let $p, q, r \in \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of positive real numbers. Use Cauchy-Schwarz inequality to show that

   $$(p + q + r)\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) \geq 9$$

   appropriately choosing $\underline{x}$ and $\underline{y}$.

   (ii) Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Show that

   $$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 \leq \frac{1}{n}\sum_{k=1}^{n} a_k^2$$

   appropriately choosing $\underline{x}$ and $\underline{y}$.

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2. a) Sketch the level curves of the function $h(x, y) = 4x^2 + y^2 = k$ when $k = 0.25, 0.5, 0.75, 1$.

   b) (i) Use the $\epsilon - \delta$ definition of the limit to verify that $\lim\limits_{(x,y)\to(1,2)} x + y = 3$.

   (ii) Use the $\epsilon - \delta$ definition to show that the function

   $$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}; & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

   is continuous at the origin.

   c) Consider the function $f(x, y) = 2x^2 + 3xy - y^2$.

   (i) Compute $f_x(2, 1)$ that is the first partial derivative of $f(x, y)$ with respect to x at the point $(2, 1)$.

(ii) Find the curve that is the intersection of the graph of $z = f(x, y)$ and the plane $y = 1$ and compute the slope of the tangent to that curve at $x = 2$.

(iii) Give a geometric interpretation of $f_x(2, 1)$.

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3. a) Let $\underline{F} = (F_1, F_2, \ldots, F_n) : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be a vector-valued function defined on an open subset $U \subseteq \mathbb{R}^m$. Define the following:

(i) $\underline{F}$ is differentiable at $\underline{a} \in U$,

(ii) the derivative of $\underline{F}$ at $\underline{a}$.

b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$. Explain clearly what you mean by the 'linear approximation of f at (a, b) denoted by $L_{(a,b)}(x, y)$' and show also that geometrically the linear approximation represents the equation of the tangent plane to the graph of f at (a, b).

Compute the equation of the plane tangent to the graph of the function

$$f(x, y) = \arctan(xy)$$

at the point $(1, 1)$.

c) Let $\underline{F} : \mathbb{R}^2 \to \mathbb{R}^3$ and $\underline{G} : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$\underline{F}(x, y) = (x^3 + y, e^{xy}, 2 + xy) \text{ and}$$
$$\underline{G}(u, v, w) = (u^2 + v, uv + w^3)$$

respectively.

Using the chain rule for functions of several variables, compute $D(\underline{G} \circ \underline{F})(0, 1)$. All the notations have usual meanings.

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4. a) Let $z = f(x, y)$ be a function of class C^2.

(i) Explain what you mean by $\underline{x}_0 = (x_0, y_0)$ is a critical point of $z = f(x, y)$.

(ii) Explain clearly the difference between the two statements that "the function $z = f(x, y)$ has a relative (local) maximum at (x_0, y_0)" and "the function $z = f(x, y)$ has a absolute (global) maximum at (x_0, y_0)".

(iii) Find the dimensions that is length, width and height of a rectangular box with a lid having a volume of 100 cm^3 and the smallest possible surface area.

b) A rectangular box without a lid is to be made from 12 m^2 of cardboard. Use the Lagrange multiplier method to find the maximum volume of such a box.
