UNIVERSITY OF RUHUNA
FACULTY OF SCIENCE
Bachelor of Science Special Degree in Mathematics
Level I (Semester I) Examination, August 2017

SUBJECT: MATHEMATICS **COURSE UNIT: MSP 3184 – MEASURE THEORY**

INSTRUCTIONS:

- Show all work, simplify your answers and write out your work neatly for full credit.
- Answer **ALL** questions in **PART I**.
- Choose and answer **ONLY FOUR** questions in **PART II**.
- Time Allowed: **THREE** hours.

PART I

1. Let $f:[0,1]\to\{0,1\}$ be defined by

$$f(x)=\begin{cases}1, & \text{if } x \text{ is rational;}\\ 0, & \text{if } x \text{ is irrational}.\end{cases}$$

Is f Riemann integrable? Justify your answer.

Find $(L)\int_0^1 f(x)\,dx$.

[40 points]

2. Find $m^*(N)$, where m^* is the Lebesgue outer measure on $\Re$. [40 points]

3. Let (X,Θ,μ) be a measure space, and let $\{E_n\}$ be a countable collection of measurable subsets of X. Show that $\mu\left(\bigcup_{n=1}^{\infty}E_n\right)\leq\sum_{k=1}^{\infty}\mu(E_n)$. [40 points]

4. Let $f:\Re\to\Re$ be defined by

$$f(x)=\begin{cases}x+5, & \text{if } x<-1;\\ 2, & \text{if } -1\leq x<0;\\ x^2, & \text{if } x\geq 0,\end{cases}$$

for all $x\in\Re$.
Show that f is measurable.

[40 points]

5. Give an example to show that the strict inequality may occur in Fatous lemma.

[40 points]

PART II

7. Define the *Lebesgue outer measure* $m^*(E)$ of a subset E of $\Re$ and the *Lebesgue measurable* subset of $\Re$. [10 points]

(a) Let $a, b \in \Re$ with $a < b$. Show that $m^*([a, b]) = b - a$. [30 points]
Hence, show that
(i) $m^*(I) = l(I)$, where I is a bounded interval. [15 points]
(ii) $m^*(J) = \infty$, where J is an unbounded interval. [15 points]

(b) Show that the interval (a, ∞) is *Lebesgue* measurable, where $a \in \Re$. [30 points]

8. (a) Let $\Theta := \{E \subseteq \Re \mid E \text{ is countable or } E^c \text{ is countable}\}$.
Show that Θ is a $\sigma-$algebra on $\Re$. [30 points]
Show further that Θ is the $\sigma-$algebra generated by $\{\{x\} \mid x \in \Re\}$. [20 points]

(b) What it is meant by a measurable function? [10 points]

Let (X, Θ) be a measurable space and let E be a subset of X that belongs to Θ.

Suppose that $f(x)$ and $g(x)$ are $[\infty, -\infty]$- valued measurable functions on X.

Show that $\{x \in X \mid f(x) < g(x)\} \in \Theta$. [40 points]

9. Let (X, Θ, μ) be a measure space and let f be a non-negative measurable function defined on X. Let the set function $\lambda : \Theta \to [0, \infty]$ be defined by

$$\lambda(A) = \int_A f \, d\mu \text{ for all } A \in \Theta.$$

(a) Show that λ is a measure on X. [40 points]

(b) Show further that $\int g \, d\lambda = \int g f \, d\mu$ for each of the following cases:

(i) $g = \chi_A$ (characteristic function) for some $A \in \Theta$. [10 points]
(ii) $g = \varphi$, a non-negative simple measurable function on X. [20 points]
(iii) g, a non-negative measurable function on X. [30 points]

Cts...

10. State the *Monotone Convergence Theorem.* [10 points]

Let (X, Θ, μ) be a measure space.

(a) Let φ be a simple measurable function defined on X with the standard representation, $\varphi = \sum_{i=1}^{m} \alpha_i \mu_{A_i}$, where α_i's are distinct and $A_i = \{x \in X : \varphi(x) = \alpha_i\}$ for all $i = 1, 2, \ldots, m$.

(i) Show that $\int_E \varphi \, d\mu = \sum_{i=1}^{m} \alpha_i \mu(E \cap A_i)$, where $E \in \Theta$. [10 points]

(ii) Hence, show that $\int_{E \cup F} \phi \, d\mu = \int_E \phi \, d\mu + \int_F \phi \, d\mu$, where E and F are mutually disjoint measurable subsets of X. [10 points]

(b) Show that

(i) $\int (\varphi + \phi) \, d\mu = \int \varphi \, d\mu + \int \phi \, d\mu$, where φ and ϕ are simple measurable functions defined on X. [30 points]

(ii) $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$, where f and g are non-negative measurable functions defined on X. [20 points]

(iii) $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$, where f and g are integrable functions defined on X. [20 points]

11. (a) State the *Fatou's Lemma.* [10 points]

Let (X, Θ, μ) be a measure space. Let f_n be a sequence of non-negative measurable functions which converges pointwise to f.

Show that if $f_n \le f$ for $n = 1, 2, \ldots$, then $\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$. [30 points]

(b) State and prove the *Lebesgue Dominated Convergence Theorem.* [30 points]

Evaluate $\lim_{n \to \infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n} dx$ justifying any interchange of limits you use. [30 points]