University of Ruhuna
Bachelor of Science General Degree
Level II (semester I) Examination – September 2017

Subject: Mathematics
Course Unit: MAT211β (Linear Algebra)
Time: Two (02) hours

Answer 04 questions only

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(1) (i) Let $A = [a_{ij}]$ be a non-singular n-square matrix. Prove, in the usual notation, that

(a) $A(adj(A)) = |A| I_n$ and

(b) $|(adj(A))| = |A|^{n-1}$.

(ii) Verify above (a) and (b) of part (i) for the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}.$$

(iii) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 2 & 3 \end{pmatrix}$$

using elementary **row** operations.

(iv) Let $A = \begin{pmatrix} 3+8i & 3-5i \\ 7-3i & 4-4i \end{pmatrix}$.

Write A as the sum of a Hermition and a skew-Hermition matrix.

(2) (a) Define the normal form of a matrix A.

Find the non-singular matrices P and Q such that PAQ is in the normal form where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 6 & 10 \end{bmatrix}.$$

Hence find the rank of A.

(b) Explain what is meant by a diagonalizable matrix.

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{pmatrix}.$$

Find a matrix P such that $P^{-1}AP$ is diagonal. Verify your answer.

(3) (a) State the necessary and sufficient conditions that should be satisfied by a non-empty subset W of a vector space V to be a subspace of V.

Define a basis S for a vector space V.

` (i) Find a basis for the vector space V spanned by vectors

$w_1 = (1,1,1)$, $w_2 = (1,2,3)$, $w_3 = (3,4,5)$ and $w_4 = (2,3,8)$.

(ii) Show that vectors $v_1 = (0,1,2)$ and $v_2 = (1,2,3)$ are linearly independent and extend the set $\{v_1, v_2\}$ to a basis of R^3.

(b) Let U and W be two subspaces of the vector space R^3 defined by

$$U = \{(x, y, z) \mid x - 2y = 2z\}$$
$$W = \{(x, y, z) \mid x + 3z = y\}.$$

Find $Dim(U)$, $Dim(W)$, $\text{Dim}(U + W)$ and $Dim(U \cap W)$ and verify the dimensional theorem

$$Dim(U) + Dim(W) = \text{Dim}(U + W) + Dim(U \cap W).$$

(4) (a) A mapping T: $\mathfrak{R}^3 \rightarrow \mathfrak{R}^4$, is defined by

$$T(x, y, z) = \{(x + y - z), (z - y), 2x, (2x + 5y - 5z)\}.$$

Show that

(i) T is linear,

(ii) $\{ (0,1,1)\}$ is a basis for Ker(T) ,

(iii) $\{(1,0,2,2),(1,-1,0,5)\}$ is a basis for Im(T).

(iv) **Verify** the rank- nullity theorem.

(b) Define what is meant by an orthonormal set of vectors in an inner product space.

If $\{v_1, v_2 .. v_n\}$ is a set of linearly independent vectors in an inner product space, then an orthogonal set of vectors $\{w_1, w_2 .. w_n\}$ can be obtained by using the Gram-Schmidt process as, in the usual notation,

$$w_1 = v_1 \text{ and}$$

$$w_k = v_k - \sum_{j=1}^{k-1} \frac{< v_k, w_j >}{\|w_j\|^2} w_j.$$

Find an orthogonal basis for $\Re^3$ starting with the basis vectors given by $v_1 = (1,0,1)$, $v_2 = (1,1,0)$ and $v_3 = (0,1,1)$.

Hence find an orthonormal basis for $\Re^3$.

(5) State clearly the condition under which a system of non-homogeneous linear equation

will have $A_{m\times n} X_{n\times 1} = D_{m\times 1}$

(i) a unique solution
(ii) no solutions
(iii) Infinitely many solutions.

(a) Show that the system of linear equations

$$3x-2y+2z = a$$
$$2x + 4y + 6z = b$$
$$-x + 6y + 4z = c$$

has no solution unless $a+c = b$.

(b) Determine the values of α and β for which the system of linear equations

$$2x-4y+z = 2$$
$$x - 3y + z = 5$$
$$3x - 7y + \alpha z = \beta$$

has

(i) a unique solution
(ii) no solutions
(iii) Infinitely many solutions.

Find the solutions in both cases this linear system is consistent.

(6) (a) Define what is meant by an Eigen value and corresponding Eigen vector of a matrix A_{nxn}.

Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}.$$

(i) Show that the characteristic equation of the matrix A is given by

$$(\lambda - 1)^2 (\lambda - 7) = 0.$$

(ii) Find the Eigen value(s) of the matrix A.

(iii) Find the corresponding Eigen space of the matrix A corresponding to the Eigenvalue $\lambda = 7$.

(b) State the Cayley-Hamilton Theorem.

Let an 3×3 matrix be

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

(i) Find the Characteristic equation of A.

(ii) Verify the Cayley-Hamilton Theorem for the matrix A.

(iii) State whether A is Derogatory or Non-Derogatory giving reasons.

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