University of Ruhuna

Bachelor of Science General Degree Level II (Semester I) Examination - August 2017

Subject: Mathematics
Course Unit: MAT 212β . (Real Analysis II)

Time :Two (02) Hours

Answer 04 Questions only.
A calculator is provided.

---

1. (a) What is the difference between a sequence and a series?

(b) What is a convergent series? What is a divergent series? Give one example for each of the cases.

(c) Explain the difference between $\sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{n} a_j$.

(d) A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H meters.

(i) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels. (Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.)

(ii) Calculate the total time that the ball travels.

---

2. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with postive terms and $\sum_{n=1}^{\infty} b_n$ is known to be divergent.

(a) (i) If $a_n > b_n$ for all n, what can you say about $\sum_{n=1}^{\infty} a_n$? Why?

(ii) If $a_n < b_n$ for all n, what can you say about $\sum_{n=1}^{\infty} a_n$? Why?

(b) Prove that if $\lim_{n\to\infty} \frac{a_n}{b_n} \to \infty$, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

(c) Use Part (*b*) to show that the following series diverges.

(i) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

(d) Give an example of a pair of series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with positive terms where $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ diverges and $\sum_{n=1}^{\infty} a_n$ converges.

---

3. (a) Draw a picture to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.3}} < \int_{1}^{\infty} \frac{1}{x^{1.3}}\, dx.$$

What can you conclude about the series?

(b) Suppose f is a continuous, positive, and decreasing function for $x \geq 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_{1}^{6} f(x)\, dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i$$

(c) Find the values of p for which the following series is convergent

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p}.$$

(d) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Estimate the error in using s_{10} as an approximation to the sum of the series.

Continued.

4. Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n\to\infty} r_n = L < 1$, so $\sum_{n=1}^{\infty} a_n$ converges by the Ratio Test. As usual, let R_n be the remainder after n terms, that is,

$$R_n = a_{n+1} + a_{n+2} + \ldots$$

(a) If r_n is a decreasing sequence and $r_{n+1} < 1$, show, by summing a geometric series, that

$$R_n \leq \frac{a_{n+1}}{1 - r_{n+1}}.$$

(b) If r_n is an increasing sequence, show that

$$R_n \leq \frac{a_{n+1}}{1 - L}.$$

(c) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} \frac{1}{n\, 2^n}$. Estimate the error in using s_5 as an approximation to the sum of the series.

(d) Find a value of n so that R_n is within 0.00005 of the sum. Use this value of n to approximate the sum of the series.

---

5. (a) What is a power series?

(b) What is the radius of convergence of a power series? How do you find it?

(c) What is the interval of convergence of a power series? How do you find it?

(d) Find the radius of convergence and the interval of convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$.

(e) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(x+1)^2}.$$

Continued.

(f) Use part (e) to find a power series representation for

$$f(x) = \frac{1}{(x+1)^3}.$$

---

6. (a) Let $[a, b]$ be a closed bounded interval and let f be a function on $[a, b]$. Define each of the followings:

(i) A partition of $[a, b]$

(ii) Upper Riemann sum of f over $[a, b]$

(iii) Lower Riemann sum of f over $[a, b]$

(iv) Upper Riemann integral of f over $[a, b]$

(v) Lower Riemann integral of f over $[a, b]$

(vi) Riemann integral of f over $[a, b]$.

(b) Let f be the function

$$f(x, y) = \begin{cases} |x| & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) sketch the graph of f.

(ii) In the usual notation, evaluate $L(f, P)$ and $U(f, P)$ for the partition

$$P = [-1, -\frac{1}{2}], [-\frac{1}{2}, 0], [0, \frac{1}{2}], [\frac{1}{2}, 1].$$

(c) (i) State the Fundamental Theorem of Calculus.

(ii) Using above theorem evaluate the following:

$$\frac{d}{dx} \int_0^{x^2} \exp(-t^2)dt$$
