University of Ruhuna

Bachelor of Science General Degree (Level III) Semester I Examination - August/September 2017

Subject :Applied Mathematics
Course unit: AMT311β /MAM3113 (Numerical Analysis)

Time :Two (02) Hours

Answer four (04) Questions only

Only the calculators provided by the University are allowed to use.

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1. a) Define, in the usual notation, the matrix norms $\| A \|_1$, $\| A \|_2$, $\| A \|_\infty$ and the condition number, $\kappa(A)$ of a non singular matrix A of order n.

   Find $\| A \|_1$ and $\| A \|_\infty$ for the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & -3 \end{pmatrix}.$$

   b) Solve the following system of linear equations using Gauss elimination method.
   $2x + 4y - 6z = -8$
   $x + 3y + z = 10$
   $2x - 4y - 2z = -12$

   c) Apply Doolittle method and solve the following system of linear equations.

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}.$$

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2. a) The system of linear equations $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, has an equivalent representation of the form $x = Tx + c$, where $T \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}^n$.
   Suppose that the system has a unique solution $x^* \in \mathbb{R}^n$.
   Consider the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ generated by the recurrence formula $x^{(k+1)} = Tx^{(k)} + c$, where $k = 0, 1, 2, ...$ with the initial approximation $x^{(0)}$.
   Show that $x^{(n)} - x^* = T^n(x^{(0)} - x^*)$.

Continued.

b) (i) Consider the system of linear equations $Ax = b$ and the decomposition $A = L + D + U$, where L, D and U represent the lower triangular, diagonal and upper triangular parts of A respectively.
Use this decomposition to formulate Jacobi iteration method for solving $Ax = b$. Write down the general formula of this iteration method.

(ii) Consider the system of linear equations given by

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 8 \end{pmatrix}.$$

Taking the initial approximation as $x^{(0)} = (0\ 0\ 0)^T$, find the second iterate $x^{(2)}$ using the jacobi iteration method.

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3. a) Consider the rectangular region $D = \{(t, y) | 0 \leq t \leq 5,\ -1 \leq y \leq 1\}$.
Let $f(t, y) = t^2 y - 1$ with $y(0) = 1$.
Does f satisfy the Lipschitz condition on D ? If so find a Lipschitz constant.

b) Consider the initial value problem $y' = xe^y$; $\quad y(0) = 0$.
Calculate the Picard Iterations $y_1(x)$ and $y_2(x)$ for this initial value problem.

c) Consider the initial value problem $y''(x) + 2y'(x) - 3y(x) = 6x$, with $y(0) = 0$ and $y'(0) = 1$.

(i) Transform this initial value problem into an equivalent system of first order differential equations.

(ii) Apply Euler's explicit method to the system with step size 0.2, and find the approximate values for y' and y at $x = 0.2$ and $x = 0.4$.

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4. a) Describe the Predictor-Corrector technique in approximating the solution of an initial value problem using modified Euler method and Euler's explicit method.
Use the Predictor-Corrector method described above to the initial value problem $y' = 2y/x$; $y(1) = 2$ with step size $h = 0.25$ to obtain the approximate value of $y(1.25)$.

b) In the usual notation, write down the general form for the fourth order Runge Kutta method for solving the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ with step size h.

The fourth order classical Runge-Kutta method is described by the following Butcher's table.

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 2/6 & 2/6 & 1/6 \end{array}$$

Continued.

Write down the corresponding Runge-Kutta scheme.
Apply this scheme to the initial value problem $y' = x^2 + y^2, \quad y(0) = 0$, with the step size 0.2 to obtain the approximate value of $y(0.4)$.

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5. Let $u(x,t)$ be the solution of the heat equation, $u_{xx} = cu_t$ for $t > 0$ and $0 < x < a$ with boundary conditions $u(0,t) = T_0$ and $u(a,t) = T_1$, for $t > 0$ and, initial condition $u(x,0) = f(x)$, for $0 \leq x \leq a$.

   Consider the discretization $x_i = ih$, for $i = 0, 1, 2, ..., n$ and $t_j = jk$, for $j = 0, 1, 2, ...$ where $h = \frac{a}{n}, k > 0$ are sufficiently small step sizes in x and t directions respectively.

   Derive, in the usual notation, the explicit finite difference scheme
   $u_{i,j+1} = ru_{i+1,j} + (1-2r)u_{i,j} + ru_{i-1,j}$; where $r = \frac{k}{ch^2}$.

   Draw the stencil for the scheme.

   Consider the heat equation $2u_{xx} = u_t$ for $0 < t < 1.5$ and $0 < x < 4$ with boundary conditions $u(0,t) = u(4,t) = 0$, for $0 \leq t \leq 1.5$ and the initial condition $u(x,0) = 50(4-x)$, for $0 < x < 4$.

   Using the above explicit finite difference scheme with $h = 1$ and $k = 0.25$, solve the heat equation $2u_{xx} = u_t$ for $0 < t < 1.5$ and $0 < x < 4$.

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6. Let $u(x,t)$ be the solution of the wave equation, $u_{tt} = c^2 u_{xx}$ for $0 < x < a$ and $t > 0$ subject to the initial conditions $u(x,0) = f(x)$ and $\dfrac{\partial u}{\partial t}(x,0) = g(x)$, for $0 \leq x \leq a$ and boundary conditions $u(0,t) = \phi(t)$ and $u(a,t) = \phi(t)$, for $t \geq 0$.

   Consider the discretization $x_i = ih$, for $i = 0, 1, 2, ..., n$ and $t_j = jk$, for $j = 0, 1, 2, ..., m$ where $h = \frac{a}{n}, k = \frac{b}{m}$ are sufficiently small step sizes in x and t directions respectively.

   Derive, in the usual notation, the explicit finite difference scheme
   $u_{i,j+1} = -u_{i,j-1} + \alpha^2(u_{i+1,j} + u_{i-1,j}) + 2(1-\alpha^2)u_{ij}$; where $\alpha = \frac{ck}{h}$.

   Consider the wave equation, $u_{tt} = 4u_{xx}$ for $0 < x < 5$ and $0 < t < 2$ with the initial conditions $u(x,0) = x(5-x)$ and $\dfrac{\partial u}{\partial t}(x,0) = 0$, for $0 \leq x \leq 5$ and boundary conditions $u(0,t) = u(5,t) = 0$, for $0 \leq t \leq 2$.

   Using the explicit finite difference scheme derived above, solve this wave equation with $h = 1$ and $k = 0.5$.
