University of Ruhuna

Bachelor of Science General Degree
Level III (Semester I) Examination

August 2017

Subject: Mathematics
Course Unit: MAT 311β /MPM 3113 (Group Theory)
Time: Two (02) Hours

Answer Four (04) Questions only

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1. a) Consider the set $A = \mathbb{R} \times \mathbb{Z}$ with the operation $*$ defined as $(x, n) * (y, m) = (x + 2^n y, n + m)$; where $x, y \in \mathbb{R}$ and $n, m \in \mathbb{Z}$.
Prove that $(A, *)$ is a group.
Is $(A, *)$ an abelian group? Justify your answer.

b) Let $S = \mathbb{N} \cup \{0\}$ and let $\circ$ be the binary operation defined on S by $x \circ y = |x - y|$, for all $x, y \in S$.
Does S form a group under the operation $\circ$? Justify your answer.

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2. a) Show that a necessary and sufficient condition that a non-empty subset H of a group G to be a subgroup is $a, b \in H \Rightarrow ab^{-1} \in H$.

b) Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc \neq 0 \right\}$
be a group under matrix multiplication.

(i) Write the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

(ii) Prove that $H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a = \pm 1,\ b \in \mathbb{Z} \right\}$ is a subgroup of G.

c) Let G be an abelian group. Prove that $H = \{x \in G \mid x = x^{-1}\}$ is a subgroup of G.

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3. a) Let $G = \{1, 3, 7, 9\}$ and for $a, b \in G$ the operation $\otimes_{10}$ defined by $a \otimes_{10} b = r,\ 0 \leq r < 10$; where r is the remainder when ordinary multiplication ab is divided by 10.

(i) Show that $(G, \otimes_{10})$ is a group.

(ii) Find the order of each element in G.

(iii) Is G cyclic? Justify your answer using part (ii).

b) (i) Show that if a cyclic group G is generated by an element a of order n, then a^m is a generator of G if and only if g.c.d.$(m, n) = 1$.

(ii) Write the elements of $\mathbb{Z}_{11}^*$, the set of non-zero integers modulo 11.

(iii) It is given that 2 is a generator of the group $(\mathbb{Z}_{11}^*, \otimes_{11})$. Using part (i), find all the other generators of $(\mathbb{Z}_{11}^*, \otimes_{11})$.

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4. Let G be a group and H be a subgroup of G. Show that if Ha and Hb are two right cosets of H in G, then either $Ha \cap Hb = \phi$ or $Ha = Hb$.

a) (i) Express the permutation $\rho = (1326)(124)(35)$ as a single cycle or as a product of disjoint cycles.

(ii) Find $o(\rho)$.

b) Let $\tau = \alpha^{-1}\beta^2$, where $\alpha = (123), \beta = (5432)$.

(i) Find the permutation τ.

(ii) Is τ an even permutation or an odd permutation? Justify your answer.

c) Let $H = \{I, (123), (132)\}$ be a subgroup of S_3; where $S_3 = \{I, (12), (13), (23), (123), (132)\}$ is a group under composition of permutations. Show that H is normal in S_3 by listing all its left and right cosets.

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5. a) Let f be a mapping from $(\mathbb{Z}, +)$ to the group $G = \{1, -1\}$ under multiplication defined as

$$f(x) = \begin{cases} 1 & ; x \text{ is even}, \\ -1 & ; x \text{ is odd}. \end{cases}$$

Show that $f : \mathbb{Z} \to G$ is a homomorphism.
Is $f : \mathbb{Z} \to G$ an isomorphism? Justify your answer.

b) Let G, G' be two groups and $f : G \to G'$ be a homomorphism.

(i) Define the kernel of f (Ker f).

(ii) Prove that f is one-one if and only if Ker $f = \{e\}$, where e is the identity element of G.

(iii) Let $R = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \mid x, y, z \in \mathbb{C} \right\}$ be a group under matrix addition and $S = \{(x, y) | x, y \in \mathbb{C}\}$ be a group under addition. Define $\theta : R \to S$ such that $\theta\left[\begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \right] = (x, y)$.
Show that θ is a homomorphism.
Find Ker θ.

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6. a) For $a, b \in \mathbb{R}, a \neq 0$, define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi_{ab}(x) = ax + b$.
Let $G = \{\phi_{ab} \mid a, b \in \mathbb{R}, a \neq 0\}$ and $N = \{\phi_{ab} \in G \mid a = 1, b \in \mathbb{R}\}$.
Prove that N is a normal subgroup of G.

b) Let $f : G \to G'$ be an onto homomorphism and let $K = \text{Ker} f$.
For H', a subgroup of G', define $H = \{x \in G \mid f(x) \in H'\}$.
Show that

(i) H is a subgroup of G.

(ii) $K \subseteq H$.

(iii) If H' is normal in G', then H is normal in G.
