University of Ruhuna

Bachelor of Science General Degree Level III (Semester I) Examination - August 2017

Subject: Mathematics
Course Unit: MAT312β . (Real Analysis III)

Time :Two (02) Hours

Answer 04 Questions only.

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1. (a) Let $\mathbf{a} \in \mathbb{R}^n$, $r > 0$, and $\mathbf{A}$ be a subset of $\mathbb{R}^n$. Define the following terms:

   (i) An open $n-$ball in $\mathbb{R}^n$ with the center $\mathbf{a}$ and the radius $r > 0$.

   (ii) An interior point of $\mathbf{A}$.

   (iii) An exterior point of $\mathbf{A}$.

   (b) Prove the Cauchy-Schwartz inequality

   $$|\mathbf{x}.\mathbf{y}| \leq \| \mathbf{x} \| . \| \mathbf{y} \|, \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

   (c) Using the Cauchy-Schwartz inequality prove that the triangle inequality

   $$\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|, \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

   (d) If x, y and z are three positive real numbers such that $x + y + z \leq 3$ then show that

   $$\tfrac{1}{x} + \tfrac{1}{y} + \tfrac{1}{z} \geq 3.$$

   (e) Determine whether each of the following set is open or not. Graph each of the set.

   (i) $\mathbf{A} = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 3\}$

   (ii) $\mathbf{B} = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \;\; y \geq 0\}$.

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2. Let $f(x, y)$ be a two variable function defined on a subset $\mathbf{A}$ of $\mathbb{R}^2$.

(a) Define the first partial derivatives $f_x(a, b)$ and $f_y(a, b)$ of f at the point $(a, b) \in \mathbb{R}^2$.

(b) Define the differentiability of $f(x, y)$ at the point $(x, y) \in \mathbb{R}^2$.

(c) Let

$$f(x, y) = \begin{cases} \frac{xy\,(x^2 - y^2)}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

(i) Find $f_x(0, 0)$ and $f_y(0, 0)$.

(ii) Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

(iii) Check the differentiability of $f(x, y)$ at the point $(0, 0)$.

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3. Let $f : \mathbf{A} \to \mathbb{R}$ be a function defined on an open set $\mathbf{A} \subset \mathbb{R}^n$.

(a) (i) What is meant by $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$.

(ii) State the (ϵ, δ) definition for the continuity of the function f at the point $\mathbf{x}_0 \in \mathbb{R}^n$.

(b) Find the following limits:

(i) $$\lim_{(x,y) \to (0,0)} \frac{\ln(x^2 + y^2 + 1)}{x^2 + y^2},$$

(ii) $$\lim_{(x,y) \to (1,2)} \frac{\sin^{-1}(xy - 2)}{\tan^{-1}(3xy - 6)}.$$

Continued.

(c) Let

$$f(x,y) = \begin{cases} 1 & xy \neq (0,0) \\ 0 & xy = (0,0). \end{cases}$$

Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist.

(d) Use part (a), to show that

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

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4. (a) State clearly, the conditions of Schwarz's theorem and Young's theorem for the equality of $f_{xy}(x,y)$ and $f_{yx}(x,y)$ of the two variable function $f(x,y)$.

(b) Let

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(i) Show that $f_{xy}(0,0) = f_{yx}(0,0)$.

(ii) Show that $f(x,y)$ does not satisfy the conditions of Schwarz's theorem and also of Young's theorem at the origin.

Continued.

5. (a) Explain briefly, how you find the extreme values of a two-variable function $f(x, y)$ defined on a subset in $\mathbb{R}^2$.

(b) Write down a necessary condition for $f(x, y)$ to have an extreme value at (a, b).

(c) Find the maxima and minima of the function

$$f(x, y) = 6x - 4y - x^2 - 2y^2.$$

(d) Verify the Jacobian property

$$\frac{\partial(fg, h)}{\partial(u, v)} = \frac{\partial(f, h)}{\partial(u, v)} g + \frac{\partial(g, h)}{\partial(u, v)} f,$$

where f, g and h are differentiable functions of the variables u and v.

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6. (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function at (a, b, c). Show that

(i) the tangent plane to the surface $f(x, y, z) = 0$ at the point (a, b, c) is given by

$$f_x(a, b, c)\ (x - a) + f_y(a, b, c)\ (y - b) + f_z(a, b, c)\ (z - c) = 0. \text{ and}$$

(ii) the normal line to the surface at (a, b, c) is given by

$$\frac{(x - a)}{f_x(a, b, c)} = \frac{(y - b)}{f_y(a, b, c)} = \frac{(z - c)}{f_z(a, b, c)}.$$

(b) For the surface given by the equation $f(x, y, z) = x^3 - y^3 - 2\,x\,y + 4\,x^2\,y + z$, find the equations of the

(i) tangent plane

(ii) normal line

at the point $(1, 1. -1)$ on the surface.

(c) Minimize $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ subject to $x + y + z + w = 10$ and x - y +z + 3w = 6.
