University of Ruhuna

Bachelor of Science (General) Degree Level I
Semester II Examination

January 2018

Subject: Applied/ Industrial Mathematics
Course Unit: AMT/IMT122β (Mathematical Modelling I)

Time: Two (02) Hours

Answer <u>04 Questions</u> only.

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1. a) Suppose in a chemical reaction two substances, A and B, react in equal amounts to form a compound C. Let $u(t)$ be the concentration of the compound C at time t, which satisfies the dif ferential equation

$$\frac{du}{dt} = r(a-u)(b-u)$$

where r is a positive constant; a and b are initial concentrations of A and B at time $t = 0$. Assuming $u(0) = 0$, obtain the concentration u as a function of time. Also, determine the limiting concentration.

b) A spherical raindrop of radius a falls from a height h and accumulates moisture from the atmosphere as it descends, thereby increasing the radius, r, of the spherical raindrop at a rate λa. That is $\frac{dr}{dt} = \lambda a$. Let $M(t)$ and $v(t)$ be the mass and velocity of the raindrop at time t. Derive the equation of the motion of the raindrop in the form $\frac{d}{dt}(Mv) = Mg$. Show the followings

(i) $v = \frac{g}{4\lambda}\left(\frac{r}{a} - \frac{a^3}{r^3}\right)$.

(ii) $\frac{dx}{dr} = \frac{g}{4a\lambda^2}\left(\frac{r}{a} - \frac{a^3}{r^3}\right)$, where x is the fallen distance of the raindrop up to time t.

(iii) When it reaches the ground it's radius is

$$\lambda a\sqrt{\frac{2h}{g}}\left(1 + \sqrt{1 + \frac{g}{2h\lambda^2}}\right).$$

Continued.

2. a) Suppose a savings account is opened with initial deposit of C Rupees that pays $r\%$ interest compounded yearly and w Rupees is withdrawn at the end of each year.

(i) If a_n is the balance of the account at the end of year n $(n = 0, 1, 2, 3, ...)$, then show that the mathematical model for this process is given by

$$a_{n+1} = (1 + r)a_n - w$$

where $a_0 = C$.

(ii) Find the equilibrium value, a^*, of the model.

(iii) derive an expression for a_{n+1} in terms of C, r and w.

(iv) Explain the long term behaviour of the account balance for the cases $C = a^*$, $C < a^*$ and $C > a^*$ separately.

(v) If $C = 50000$, $r = 10\%$ and $w = 1000$ then find the account balance at the end of the fifth year.

b) Initially $25kg$ of salt is dissolved in a large tank holding $1800l$ of water. Brine solution is pumped into the tank at a rate of $12l$ per minute, and a well-stirred solution is then pumped out at the same rate. The concentration of the solution entering is $1kg$ per $4l$.

(i) Show that the following ordinary differential equation

$$\frac{dA}{dt} = 3 - \frac{A}{150}$$

can be proposed to model the above situation, where A is the amount of salt (in kgs) in the tank at time t.

(ii) Solve the above differential equation and determine the amount of salt in the tank after a long time.

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3. a) State the Newton's law of cooling and write down relevant differential equation. A body with initial temperature $300°C$ is placed in a large block of ice. Temperature of the body after 10 minuets is $75°C$.

(i) Obtain an expression for the temperature T of the body as a function of time t.

(ii) Find the temperature of the body at $t = 5$ and $t = 20$ minuets.

(iii) Find the time when it's temperature reaches $38.5°C$.

b) Suppose a student carrying a flue virus returns to an isolated collage campus of 1000 students. Assume that the rate at which the virus spreads is proportional not only to the number, x, of the infected students but also to the number of students not infected. Also, assume that no one leaves the campus throughout the duration of the disease

(i) Construct a continuous mathematical model for this situation in the form

$$\frac{dx}{dt} = kx(1000 - x),$$

where k is a constant.

Continued.

(ii) Find the solution of the model you derived in section $b(i)$ if it is further observed that after 10 days the infected number of students is 50.

(iii) Find the limiting behaviour of the number of infected students.

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4. a) Consider the following linear system of ordinary differential equations

$$\frac{dx}{dt} = 2x + 7y,$$
$$\frac{dy}{dt} = -5x - 10y.$$

(i) Determine the stability behaviour of the equilibrium point.

(ii) Sketch the phase plane diagram.

(iii) If $x(0) = 2$ and $y(0) = 0$ then find the exact solution of the above system.

b) Consider the following linear system of ordinary differential equations

$$\frac{dx}{dt} = x + 2y - 6,$$
$$\frac{dy}{dt} = 6x - 3y + 24.$$

(i) Find the equilibrium point of the system

(ii) Determine the stability behaviour of the equilibrium point.

(iii) Sketch the phase plane diagram about the equilibrium point.

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5. Consider the following nonlinear system of ordinary differential equations

$$\frac{dx}{dt} = 2x + x^2 - xy,$$
$$\frac{dy}{dt} = xy - 2y.$$

a) Find the equilibrium points of the system.

b) Determine the stability of the equilibria.

c) Sketch the phase plane diagram of the system.

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6. a) In a certain forest $r\%$ of the trees are destroyed naturally each year. Also, H number of trees are harvested for timber (cut down for industrial use) but P number of new trees are either planted or sprout up on their own. Here, H and P are constants. Let T_n be the number of trees at the end of n^{th} year.

(i) Formulate a discrete mathematical model for this process.

(ii) Find an expression for T_n in terms of T_0, r, H and P.

Continued.

(iii) What is the number of trees in the forest will have in 5 years, if $r = 3\%$, $H = 4000$ and $P = 8000$ and the currently estimated number of trees in that forest be $T_0 = 200000$?

b) We consider a forest containing tigers (predator) and deer (prey). Let T_n and D_n be the respective population of tigers and deer at the end of year n. We assume that deer are the only source of food for the tigers and the tigers are the only predators for the deers.

The tiger population decreases at a rate α if no deers are available and grows at rate β when the food (deer) is available. Similarly, the deer population grows at a rate γ when no tigers are around and decreases at a rate δ in the presence of tiger population

(i) Formulate a discrete mathematical model for this process.

(ii) In the usual notation, show that the corresponding Jacobian matrix, A, is given by

$$A = \begin{pmatrix} 1 - \alpha & \beta \\ -\delta & 1 + \gamma \end{pmatrix}.$$

(iii) Find the eigenvalues of A.

(iv) Determine the stability of the equilibrium point if $\alpha = 0.5$, $\beta = 0.4$, $\gamma = 0.1$ and $\delta = 0.17$.

(v) Discuss the stability of the equilibrium point if δ is changed to 0.05.
