University of Ruhuna

Bachelor of Science (Special) Degree
Level II (Semester II) Examination
February 2018

Subject: Mathematics
Course Unit: MSP4224 (Introduction to Stochastic Analysis with Applications)
Time: Three (03) Hours

Answer **ALL Questions.**

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1. a) Let $\mathcal{M} = \{B_i,\ i = 1, 2, \ldots, 6\}$ be a partition of the sample space S and define a sigma algebra $\mathcal{E}$ generated by $\mathcal{M}$ as $\mathcal{E} = \sigma(\mathcal{M})$. Now, define a probability measure P on $\{S, \mathcal{E}\}$ by

$$P(B_1) = \frac{1}{4},\ P(B_2) = \frac{1}{4},\ P(B_3) = \frac{1}{8},\ P(B_4) = \frac{1}{8},\ P(B_5) = \frac{1}{6},\ P(B_6) = \frac{1}{12},$$

and a random variable Z by

$$Z(\omega) = i,\ \text{if } \omega \in B_i,\ i = 1, 2, \ldots, 6.$$

Furthermore, let $C_1 = B_1 \cup B_2, C_2 = B_3 \cup B_4, C_3 = B_5 \cup B_6$ and define $\mathcal{N}$ to be the partition $\mathcal{N} = \{C_1, C_2, C_3\}$. Finally, let $\mathcal{F} = \sigma(\mathcal{N})$.

Compute the conditional expectation $E[Z|\mathcal{F}]$.

b) Consider the stochastic process $Z = (Z_n,\ n = 0, 1, 2, \ldots)$. State the conditions for Z to be a discrete-time martingale with respect to the filtration $(\mathcal{F}_n,\ n = 0, 1, 2, \ldots)$.

Suppose that $X_1, X_2, \ldots$ are independent identically distributed random variables with

$$P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}.$$

Let $S_0 = 0$ and define $S_n = X_1 + X_2 + \cdots + X_n$ for $n = 1, 2, \ldots$. Then the sequence $\{S_n, n = 0, 1, 2, \ldots\}$ is a simple random walk starting at zero.

Show that

(i) $\{M_n, n = 0, 1, 2, \ldots\}$, where $M_n = S_n^2 - n$, is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(M_0, M_1, \ldots, M_n) = \sigma(S_0, S_1, \ldots, S_n)$.

(ii) $\{R_n, n = 0, 1, 2, \ldots\}$, where $R_n = (\operatorname{sech}\theta)^n \exp(\theta S_n); n = 0, 1, 2, \ldots$, is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(R_0, R_1, \ldots, R_n) = \sigma(S_0, S_1, \ldots, S_n)$.

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2. a) Define the standard Brownian motion $(B_t,\ t \geq 0)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(B_t, t \geq 0)$ be a one-dimensional Brownian motion. Prove that

(i) for any t, $E(B_t) = 0$ and $E(B_t^2) = t$,

(ii) for any $s, t \geq 0$, $E(B_s B_t) = \min(s, t)$,

where E denotes expectation.

b) Let X_t and Y_t be two stochastic processes. State and prove the product rule for stochastic differentiation.

Apply the product rule to show the following. Here, B denotes Brownian motion.

(i) $d(B_t^2) = 2B_t\, dB_t + dt$,

(ii) $d(B_t^3) = 3B_t^2\,dB_t + 3B_t dt$,

(iii) Let $Z_t = \int_0^t B_u\,du$ be the integrated Brownian motion, show that $dZ_t = B_t\,dt$,

(iv) Let $A_t = \frac{Z_t}{t}$ be the average of the Brownian motion on the time interval $[0, t]$. Show that

$$dA_t = \frac{1}{t}\left(B_t - \frac{1}{t}Z_t\right) dt,$$

c) Using the generalized Ito lemma, show that it is the function $g(t, x) = \exp\{x - \frac{1}{2}t\}$ gives rise to the Ito exponential function, not the usual function $g(x) = e^x$.

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3. a) Consider the stochastic process $F_t = f(t)g(B_t)$, where f and g are differentiable functions. Using the product rule and the simple Ito lemma, obtain the integration by parts formula

$$\int_a^b f(t)\,g'(B_t)\,dB_t = f(t)g(B_t)\Big|_a^b - \int_a^b f'(t)g(B_t)\,dt - \frac{1}{2}\int_a^b f(t)g''(B_t)\,dt$$

on $[a, b]$.

Let $f(t) = e^{\alpha t}$, where α is a constant and $g(x) = \cos x$. Compute the stochastic integrals

$$\int_0^T e^{\alpha t}\cos B_t\,dB_t \text{ and } \int_0^T e^{\beta t}\sin B_t\,dB_t.$$

Obtain the results for the particular cases $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$.

Hence and using the Euler's formula, deduce that

$$\int_0^T \exp\left\{\frac{t}{2} + iB_t\right\} dB_t = i\left(1 - \exp\left\{\frac{T}{2} + iB_T\right\}\right).$$

b) Let the Ito process $X(t)$ have stochastic differential

$$dX(t) = X(t)\,dB(t) + \frac{1}{2}X(t)\,dt.$$

Applying simple Ito lemma for $\ln X(t)$, find a process $X(t)$ satisfying the above stochastic differential.

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4. a) Show that the stochastic process

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}\,dB_s,\ 0 \le t < 1,\ a, b \in \mathbb{R}$$

is solution of the stochastic differential equation

$$dY_t = \frac{b - Y_t}{1-t}\,dt + dB_t,\ 0 \le t < 1,\ Y_0 = 0.$$

You may use the result

$$d\left(\int_0^t \frac{1}{1-s}\,dB_s\right) = \frac{1}{1-t}\,dB_t$$

if necessary.

b) Solve the stochastic differential equation

$$dZ_t = t^2 dt + \exp\left(\frac{t}{2}\right)\cos B_t\,dB_t,\ Z_0 = 0,$$

and find the expectation, $E[Z_t]$ and variance $\text{Var}(Z_t)$.