UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 2 Examination in Engineering: February 2020

Module Number: IS2401 **Module Name: Linear Algebra & Differential Equations**

[Three hours]

[Answer **all questions,** each question carries **12** marks]

Q1. a) The temperature of a body is changing from 100°C to 80°C in 10 minutes. Find the time when the temperature will be 50°C, if the temperature of the air is 30°C. Assume that the rate of change of the temperature is given by

$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \;\; k > 0,$$

where $\theta(t)$ denote the temperature of the body at a time t, θ_0 is the temperature of the surrounding medium and k is the constant of proportionality.

[3 Marks]

b) Determine for what values of a and b, the following differential equation is exact and obtain the general solution of the exact equation

$$(y + x^3)dx + (ax + by^3)dy = 0.$$

[2 Marks]

c) Prove that

i. if $F(\alpha) \neq 0$, then $\frac{1}{F(D)}\{e^{\alpha x}\} = \frac{1}{F(\alpha)}e^{\alpha x}$.

ii. $\frac{1}{1-D}\{x^n\} = \sum_{r=0}^{n} \frac{n!}{(n-r)!}x^{n-r}$.

[2 Marks]

d) Obtain the solution of the following differential equation about $x = 0$.

$$(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x + x^2$$

[5 Marks]

Q2. a) i. Define an irrotational motion.

ii. Show that $\mathbf{F} = y(z - 2x)\mathbf{i} - x(x - z)\mathbf{j} + xy\mathbf{k}$ is an irrotational motion and hence, find the scalar potential of $\mathbf{F}$.

[3 Marks]

b) Evaluate $\int_C \mathbf{F}.d\mathbf{r}$ along the following path C, where $\mathbf{F} = 2y\mathbf{i} + (xz+3)\mathbf{j} + (yz - x)\mathbf{k}$.

i. $x = t^2,\ y = 2t^2,\ z = t$ from $t = 0$ to $t = 2$.

ii. The polygonal path consisting of the line segments from (0,0,0) to (0,0,1), from (0,0,1) to (0,1,1) and from (0,1,1) to (2,1,1).

iii. The line segment joining from (0,0,0) to (2,2,1).

[3 Marks]

c) Evaluate $\int_S \mathbf{A}.\mathbf{n}ds$, where $\mathbf{A} = xy\mathbf{i} - x^2\mathbf{j} + (x+z)\mathbf{k}$, and S is the portion of the plane $2x + 2y + z = 6$ included in the first octant, and $\mathbf{n}$ is a unit normal to S.

[3 Marks]

d) i. Sketch the three-dimensional region $\mathcal{R}$ bounded by $x + y + z = a\ (a > 0)$, $x = 0, y = 0, z = 0$.

ii. Give a physical intepritation to

$$\iiint_V (x^2 + y^2 + z^2)\, dxdydz.$$

iii. Evaluate the triple integral in part ii).

[3 Marks]

Q3. a) State

i. The divergence theorem

ii. Stockes' theorem.

[2 Marks]

b) Verify the divergence theorem for $\mathbf{A} = (2x - z)\mathbf{i} + x^2y\mathbf{j} - xz^2\mathbf{k}$, taken over the region bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

[3 Marks]

c) Verify Stockes's theorem for $\mathbf{A} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 9$ and C is its boundary.

[3 Marks]

d) Let V be the volume bounded by the closed surface S. The vector field $\mathbf{A}$ and the scalar field $\emptyset$ are acting on the surface S. If $\mathbf{n}$ is the outward unit normal to the surface S at the point (x, y, z) and $\mathbf{r}$ is the position vector of the point (x, y, z), prove that

i. $\iiint_V (\nabla \times \mathbf{A})dv = \iint_S (\mathbf{n} \times \mathbf{A})\, ds$

ii. $\int_C \emptyset d\mathbf{r} = \iint_S (\mathbf{n} \times \nabla\emptyset)ds = \iint_S d\mathbf{S} \times \nabla\emptyset$

[4 Marks]

Q4. a) Let V be a vector space over the field F. Explain what is meant by

i. W is a subspace of V.

ii. S is a basis of V.

iii. Sum and direct sum of subspaces U and W of V.

[3 Marks]

b) State whether each of the following is true or false. Justify your answers.

i. $W = \{(x, y, z); x, y, z \in \mathbb{R},\ xyz = 0\}$ is a subspace of $\mathbb{R}^3$ under the usual addition and scalar multiplication.

ii. If $U = \{(x + x^2, 1 - x + 2x^2, 1 + x); x \in \mathbb{R}\}$ is a basis for the vector space of polynomials of degree 2.

iii. $V = \{(x, y, z); x, y, z \in \mathbb{R} \text{ and } z = 1\}$ is not a vector spapce over $\mathbb{R}$ under the ususal addition and scalar multiplication.

iv. If $U = \left\{\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}, x, y \in \mathbb{R}\right\}$ and $W = \left\{\begin{bmatrix} x & 0 \\ 0 & t \end{bmatrix}, x, t \in \mathbb{R}\right\}$, the vector space $M_{2\times2}$ is the direct sum of U and W.

[6 Marks]

c) Let U be the subspace of $\mathbb{R}^4$ generated by the set

$$S = \{(1,1,-1,-2), (1,2,0,-1), (0,1,1,1), (2,1,-3,-5)\}$$

i. Show that S is not a linearly independent set.

ii. Find a basis and the dimension of U.

iii. Extend the above basis of U to a basis of $\mathbb{R}^4$.

[3 Marks]

Q5. a) Let $T: V \to U$ be a linear transformation, where V and U are two vector spaces over the field $\mathbb{F}$

i. Briefly explain what is meant by the Kernel and Image of T.

ii. Show that the kernel is a subspace of V and Image is a subspace of U.

[3 Marks]

b) Let $T: \mathbb{R}^3 \to \mathbb{R}^5$ be a linear transformation where,

$$T(x, y, z) = (x + y,\ 2x - z,\ 2y + z,\ 2x - 4y - 3z,\ x - y - z)$$

Find bases and dimensions of the Kernel and the Image of T.

[2 Marks]

c) If a matrix A has eigenvalue λ, show that

i. A^{-1} has eigenvalue $\frac{1}{\lambda}$

ii. A^2 has eigenvalue λ^2

[2 Marks]

d) Consider the matrix

$$A = \begin{bmatrix} 2 & 6 & -6 \\ 3 & 5 & -6 \\ 3 & 6 & -7 \end{bmatrix}$$

i. Show that $P^{-1}AP = D$; where D is a diagonal matrix whose entries are eigenvalues of A and P is a square matrix with corresponding eigenvectors.

ii. Hence, find A^5.

iii. Write down the eigenvalues of A^5.

[5 Marks]