UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: October 2019

Module Number: IS3302 **Module Name: Complex Analysis and Mathematical Transforms**

[Three Hours]

[Answer **all questions**, each question carries twelve marks]

Q1. a) Find all complex solutions of the equation $z^2(1-z^2) = 16$.

[2 Marks]

b) Show that the function $f(z) = \bar{z}$ is continuous but not differentiable as $z \to 0$.

[2 Marks]

c) Discuss whether the function $f(z) = |z|^2$ is analytic everywhere or not.

[2 Marks]

d) In the usual notations, z and w are two complex numbers in Z and W planes respectively. Find the image of the infinite strip $0 < y < {}^1/_{2c}$; $c \in \mathbb{R}$ under the map $w = {}^1/_z$.

[6 Marks]

Q2. a) Suppose that $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$, u and v are real valued functions.

i State the Cauchy – Riemann equations.

ii If $f(z) = z^n$, find the Cauchy-Riemann equations in polar form.

[5 Marks]

b) Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function. If $u(x,y) = \ln(x^2 + y^2)$, find

i $v(x,y)$.

ii $f'(z)$.

[7 Marks]

Q3. a) Find the Maclaurin expansion of $\sin^2 z$ up to the powers of z^5. Hence, write down the expansion of $\cos^2 z$ up to powers of z^6.

[3 Marks]

b) Determine the nature of all singular points of the following functions.

i $f(z) = \sec\left({}^1/_z\right)$

ii $$f(z) = \frac{\cos(\pi z)}{(z-a)^2 \sin(\pi z)}$$

[4 Marks]

c) State the Cauchy's Residue theorem in the usual notations and evaluate $\oint_C f(z)dz$ if C is the circle $|z| = 4$ for each of the following functions.

i $\quad \dfrac{z+1}{z^2(z+2)}$

ii $\quad \dfrac{z}{z^2+1}$

[5 Marks]

Q4. a) Consider the Fourier Series for a function $f(t)$ of period 2π;

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

Where, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad ; n = 1,2,3,\ldots \quad , \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \quad ; n = 1,2,3,\ldots$

i Obtain the Fourier series and Fourier coefficients for functions of general period defined in the interval $(-c, c)$.

ii Find the Fourier series expansion of the function $f(t) = |t|$; $-2 < t < 2$.

[8 Marks]

b) In the usual notations, equations of the Fourier transform and inverse Fourier transform are

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \text{ and } f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega .$$

Find the Fourier transform of the function

$f(t) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$; where a is a positive constant.

[4 Marks]

Q5. a) Use the Convolution theorem to find the inverse Laplace transform of

$$F(s) = \frac{6}{s(s^2+9)}.$$

[2 Marks]

b) Consider the differential equation given by:

$y''(t) + 2y'(t) + 10y(t) = f(t)$; $y(0) = 0, y'(0) = 1, f(t) = \gamma(t)$; where $\gamma(t)$ is a unit step function.

Find $y(t)$.

[5 Marks]

c) i Use the property $Z\{f_n a^n\} = F\left(\frac{z}{a}\right)$ to obtain the z-transform of $\{f_n\} = \{na^n\}$.

ii Solve the difference equation;

$y_n - 6y_{n-1} + 9y_{n-2} = 0 \quad n = 0, 1, 2, \ldots$; $y_{-1} = 1$, $y_{-2} = 0$.

[5 Marks]

Table of Laplace Transform Pairs

	$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\lim_{T\to\infty}\int_{c-jT}^{c+jT} F(s)e^{st}ds$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-st}dt$	
transform	$f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F(s)$	
complex conjugation	$f^*(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F^*(s^*)$	
time shifting	$f(t-a) \quad t \geqslant a > 0$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$a^{-as}F(s)$	
	$e^{-at}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F(s+a)$	frequency shifting
time scaling	$f(at)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{\lvert a\rvert}F(\frac{s}{a})$	
linearity	$af_1(t) + bf_2(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$aF_1(s) + bF_2(s)$	
time multiplication	$f_1(t)f_2(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F_1(s) * F_2(s)$	frequency convolution
time convolution	$f_1(t) * f_2(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F_1(s)F_2(s)$	frequency product
delta function	$\delta(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	1	
shifted delta function	$\delta(t-a)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	e^{-as}	exponential decay
unit step	$u(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{s}$	
ramp	$tu(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{s^2}$	
parabola	$t^2u(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{2}{s^3}$	
n-th power	t^n	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{n!}{s^{n+1}}$	
exponential decay	e^{-at}	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{s+a}$	
two-sided exponential decay	$e^{-a\lvert t\rvert}$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{2a}{a^2-s^2}$	
	te^{-at}	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{(s+a)^2}$	
	$(1-at)e^{-at}$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{s}{(s+a)^2}$	
exponential approach	$1-e^{-at}$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{a}{s(s+a)}$	
sine	$\sin(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{\omega}{s^2+\omega^2}$	
cosine	$\cos(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{s}{s^2+\omega^2}$	
hyperbolic sine	$\sinh(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{\omega}{s^2-\omega^2}$	
hyperbolic cosine	$\cosh(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{s}{s^2-\omega^2}$	
exponentially decaying sine	$e^{-at}\sin(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{\omega}{(s+a)^2+\omega^2}$	
exponentially decaying cosine	$e^{-at}\cos(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{s+a}{(s+a)^2+\omega^2}$	
frequency differentiation	$tf(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$-F'(s)$	
frequency n-th differentiation	$t^nf(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$(-1)^nF^{(n)}(s)$	
time differentiation	$f'(t) = \frac{d}{dt}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$sF(s) - f(0)$	
time 2nd differentiation	$f''(t) = \frac{d^2}{dt^2}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$s^2F(s) - sf(0) - f'(0)$	
time n-th differentiation	$f^{(n)}(t) = \frac{d^n}{dt^n}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
time integration	$\int_0^t f(\tau)d\tau = (u*f)(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{s}F(s)$	
frequency integration	$\frac{1}{t}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\int_s^\infty F(u)du$	
time inverse	$f^{-1}(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{F(s)-f^{-1}}{s}$	
time differentiation	$f^{-n}(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{F(s)}{s^n} + \frac{f^{-1}(0)}{s^n} + \frac{f^{-2}(0)}{s^{n-1}} + \ldots + \frac{f^{-n}(0)}{s}$	

Table of z-Transform Pairs

$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$			$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$	ROC
transform	$x[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X(z)$	R_x
time reversal	$x[-n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X(\frac{1}{z})$	$\frac{1}{R_x}$
complex conjugation	$x^*[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X^*(z^*)$	R_x
reversed conjugation	$x^*[-n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X^*(\frac{1}{z^*})$	$\frac{1}{R_x}$
real part	$\Re e\{x[n]\}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	R_x
imaginary part	$\Im m\{x[n]\}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	R_x
time shifting	$x[n-n_0]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$z^{-n_0}X(z)$	R_x
scaling in $\mathcal{Z}$	$a^n x[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X\left(\frac{z}{a}\right)$	$\|a\|R_x$
downsampling by N	$x[Nn], N \in \mathbb{N}_0$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{N}\sum_{k=0}^{N-1} X\left(W_N^k z^{\frac{1}{N}}\right) \quad W_N = e^{-\frac{j2\omega}{N}}$	R_x
linearity	$ax_1[n] + bx_2[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$aX_1(z) + bX_2(z)$	$R_x \cap R_y$
time multiplication	$x_1[n]x_2[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{2\pi j}\oint X_1(u)X_2\left(\frac{z}{u}\right)u^{-1}du$	$R_x \cap R_y$
frequency convolution	$x_1[n] * x_2[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X_1(z)X_2(t)$	$R_x \cap R_y$
delta function	$\delta[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	1	$\forall z$
shifted delta function	$\delta[n-n_0]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	z^{-n_0}	$\forall z$
step	$u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-1}$	$\|z\| > 1$
	$-u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-1}$	$\|z\| < 1$
ramp	$nu[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
	$n^2u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
	$-n^2u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| < 1$
	$n^3u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\|z\| > 1$
	$-n^3u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\|z\| < 1$
	$(-1)^n$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z+1}$	$\|z\| < 1$
exponential	$a^n u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
	$-a^n u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-a}$	$\|z\| < \|a\|$
	$a^{n-1}u[n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{z-a}$	$\|z\| > \|a\|$
	$na^n u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
	$n^2a^n u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{az(z+a)}{(z-a)^3}$	$\|z\| > \|a\|$
	$e^{-an}u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-e^{-a}}$	$\|z\| > \|e^{-a}\|$
exp. interval	$\begin{cases} a^n & n = 0, \dots, N-1 \\ 0 & \text{otherwise} \end{cases}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
sine	$\sin(\omega_0 n)u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z\sin(\omega_0)}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > 1$
cosine	$\cos(\omega_0 n)u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z-\cos(\omega_0))}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > 1$
	$a^n\sin(\omega_0 n)u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{za\sin(\omega_0)}{z^2-2a\cos(\omega_0)z+a^2}$	$\|z\| > a$
	$a^n\cos(\omega_0 n)u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z-a\cos(\omega_0))}{z^2-2a\cos(\omega_0)z+a^2}$	$\|z\| > a$
differentiation in $\mathcal{Z}$	$nx[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$-z\frac{dX(z)}{dz}$	R_x
integration in $\mathcal{Z}$	$\frac{x[n]}{n}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$-\int_0^z \frac{X(z)}{z}dz$	R_x
	$\frac{\prod_{i=1}^{m}(n-i+1)}{a^m m!}a^m u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{(z-a)^{m+1}}$	

Note:

$$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$$