UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: August 2018

Module Number: IS3301 **Module Name: Complex Analysis and Mathematical Transforms**

[Three Hours]

[Answer **all questions**, each question carries fourteen marks]

Q1. a) i If $\cos^2 z + \sin^2 z = 1$ for all z then, utilizing complex numbers, obtain the equivalent identity for hyperbolic functions.

ii Find all the values of $(1+i)^{1/2}$.

[4 Marks]

b) i Define the continuity of a complex function $f : D \to C, D \subset C$ at a point $z_0 \in D$.

ii Discuss the continuity of the following functions at $z = 0$.

a) $f(z) = \begin{cases} \dfrac{\mathrm{Im}(z^2)}{|z|^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$ b) $f(z) = \begin{cases} \dfrac{\bar{z}}{z} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$

[5 Marks]

c) Define the terms "Mapping" and "Conformal mapping".

In the usual notations, z and w are two complex numbers represented by the points $P(x, y)$ and $Q(u, v)$ respectively in Argand diagrams in Z and W planes. Consider the mapping $w = z^2$.

i Find an expressions for x and y in terms of u and v.

ii If the point P moves along the circle $x^2 + y^2 = 2$, find the equation of the locus of Q in the W plane.

[5 Marks]

Q2. a) Explain the followings.

i A function f is analytic at a point $z_0 \in C$.

ii A function f is analytic in a region.

iii A function f is entire.

iv A function f is harmonic.

[4 Marks]

b) Discuss whether the function $f(z) = z^2 + z$ is analytic everywhere or not.

[4 Marks]

c) Consider the function $u(x, y) = x^2 - y^2 - 2y$.

i Show that the function $u(x, y)$ is harmonic.

ii Find the conjugate harmonic function $v(x, y)$ and then find $f(z)$.

[6 Marks]

Q3. a) Find the Laurent series expansions of the function,

$$f(z)=\frac{1}{(z-1)(z-2)} \; ; \; z\in C\setminus\{1,2\} \text{ around } z=0 \text{ for;}$$

i $|z|<1$

ii $|z|>2$.

[3 Marks]

b) Determine the nature of all singular points of the following functions.

i $$f(z)=\frac{z^8+z^4+2}{(z-1)^3(3z+2)^2}.$$

ii $$f(z)=\frac{z}{(z^2+4)^2}.$$

[4 Marks]

c) State the Cauchy's Integral Formula and Cauchy's Residue Theorem in the usual notation and evaluate the following integrals.

i $\oint_c \frac{1}{z^2-5z+6}dz$; where c is the unit circle $|z|=1$.

ii $\oint_c \frac{1}{z^2(z-2)(z-4)}dz$

; where c is the rectangle joining the points $(-1,-1),(3,-1),(3,1)$ and $(-1,1)$ in the complex plane.

iii $\oint_c \cot z\,dz$; where c is any circle around zero in $0<|z|<\pi$.

[7 Marks]

Q4. a) Find the followings.

i $L[(5\cos 3t-6t^3)u(t)]$

ii $L^{-1}\left[\frac{4s}{s^2+2s+5}\right]$

iii $Z^{-1}\left[\frac{z^{-4}}{z-1}+z^{-6}+\frac{z^{-3}}{z+0.5}\right]$

Where L, L^{-1} and Z^{-1} denote Laplace Transforms, Inverse Laplace Transforms and Inverse Z Transforms respectively.

[4 Marks]

b) The equation governing the build up of charge, $g(t)$, on the capacitor of an RC Circuit in Figure Q4 is $R\frac{dq}{dt}+\frac{1}{C}q=v_0$. Where v_0 is the constant d.c. voltage. Initially, the circuit is relaxed and the circuit "closed" at $t=0$ and so $q(0)=0$ is the initial condition for the charge. Use the Laplace transform method to solve the differential equation for $q(t)$. Assume the forcing term v_0 is causal.

[5 Marks]

c) Use the "Shift property" of Z Transforms to solve the second order difference equation,

$y_n-7y_{n-1}+10y_{n-2}=0$ with $y_{-1}=y(-1)=16$ and $y_{-2}=y(-2)=5$.

[5 Marks]

Q5. a) Define the followings.

i Periodic Functions.

ii Sinusoidal Functions.

[2 Marks]

b) Consider the Fourier Series for a function $f(t)$ of period 2π;

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nt+b_n\sin nt\right)$$

Where, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt \quad ; n=1,2,3,\ldots \quad , \quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt \quad ; n=1,2,3,\ldots$

i Obtain the Fourier series and Fourier coefficients for Half-Range Sine and Cosines.

ii Define the periodic function $f(t)$ for the Triangular Wave in Figure Q5. b) and then obtain the Fourier series of $f(t)$.

[7 Marks]

c) If $F(\omega)$ be the Fourier Transform of the function $f(t)$ and $f(t)$ be the Inverse Fourier Transform respectively, then in the usual notations;

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\,dt \text{ and } f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}\,d\omega.$$

Obtain, using the integral definition, the Fourier transform of the rectangular pulse in Figure Q5. c).

Then show that the Fourier integral representation of the rectangular pulse $f(t)$ is $\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega a}{\omega}e^{i\omega t}\,d\omega$.

(Note that the pulse width is $2a$)

[5 Marks]

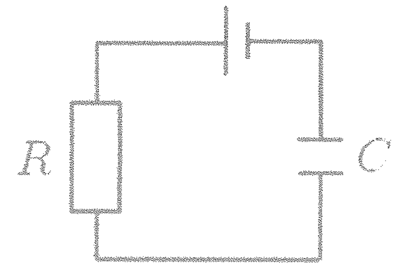

Figure Q4. : RC Circuit

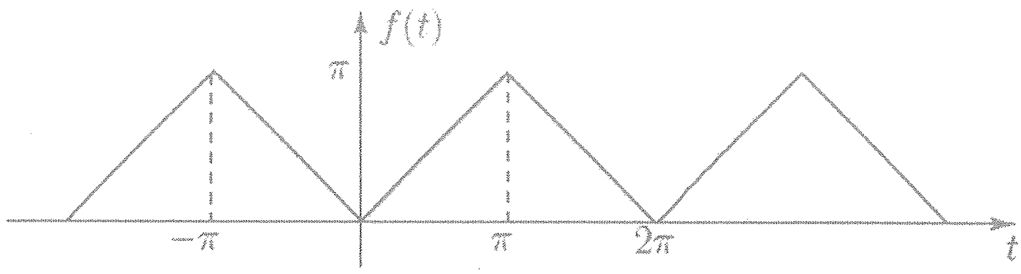

Figure Q5. b) : Triangular Wave

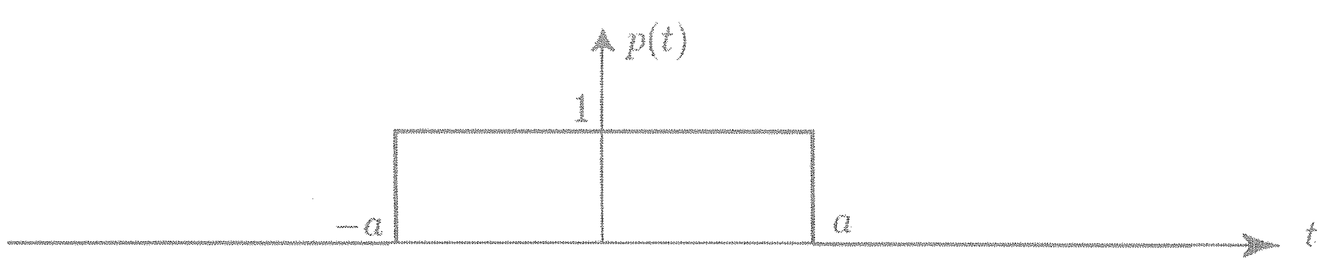

Figure Q5. c) : Rectangular Pulse

Table Q4. a) : Table of Laplace Transform Pairs

	$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{c-jT}^{c+jT} F(s)e^{st}ds$		$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-st}dt$	
transform	$f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F(s)$	
complex conjugation	$f^*(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F^*(s^*)$	
time shifting	$f(t-a) \quad t \geqslant a > 0$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$a^{-as}F(s)$	
	$e^{-at}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F(s+a)$	frequency shifting
time scaling	$f(at)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{\lvert a\rvert}F(\frac{s}{a})$	
linearity	$af_1(t)+bf_2(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$aF_1(s)+bF_2(s)$	
time multiplication	$f_1(t)f_2(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F_1(s)*F_2(s)$	frequency convolution
time convolution	$f_1(t)*f_2(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$F_1(s)F_2(s)$	frequency product
delta function	$\delta(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	1	
shifted delta function	$\delta(t-a)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	e^{-as}	exponential decay
unit step	$u(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{s}$	
ramp	$tu(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{s^2}$	
parabola	$t^2u(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{2}{s^3}$	
n-th power	t^n	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{n!}{s^{n+1}}$	
exponential decay	e^{-at}	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{s+a}$	
two-sided exponential decay	$c^{-a\lvert t\rvert}$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{2a}{a^2-s^2}$	
	tc^{-at}	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{(s+a)^2}$	
	$(1-at)c^{-at}$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{s}{(s+a)^2}$	
exponential approach	$1-e^{-at}$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{a}{s(s+a)}$	
sine	$\sin(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{\omega}{s^2+\omega^2}$	
cosine	$\cos(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{s}{s^2+\omega^2}$	
hyperbolic sine	$\sinh(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{\omega}{s^2-\omega^2}$	
hyperbolic cosine	$\cosh(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{s}{s^2-\omega^2}$	
exponentially decaying sine	$e^{-at}\sin(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{\omega}{(s+a)^2+\omega^2}$	
exponentially decaying cosine	$c^{-at}\cos(\omega t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{s+a}{(s+a)^2+\omega^2}$	
frequency differentiation	$tf(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$-F'(s)$	
frequency n-th differentiation	$t^nf(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$(-1)^nF^{(r)}(s)$	
time differentiation	$f'(t) = \frac{d}{dt}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$sF(s)-f(0)$	
time 2nd differentiation	$f''(t) = \frac{d^2}{dt^2}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$s^2F(s)-sf(0)-f'(0)$	
time n-th differentiation	$f^{(n)}(t) = \frac{d^n}{dt^n}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{(n-1)}(0)$	
time integration	$\int_0^t f(\tau)d\tau = (u*f)(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{1}{s}F(s)$	
frequency integration	$\frac{1}{t}f(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\int_s^\infty F(u)du$	
time inverse	$f^{-1}(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{F(s) \cdot f^{-1}}{s}$	
time differentiation	$f^{-n}(t)$	$\overset{\mathcal{L}}{\longleftrightarrow}$	$\frac{F(s)}{s^n}+\frac{f^{-1}(0)}{s^n}+\frac{f^{-2}(0)}{s^{n-1}}+\ldots+\frac{f^{-n}(0)}{s}$	

Table Q4. c) : Table of Z- Transform Pairs

	$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$	ROC
transform	$x[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X(z)$	R_x
time reversal	$x[-n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X(\frac{1}{z})$	$\frac{1}{R_x}$
complex conjugation	$x^*[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X^*(z^*)$	R_x
reversed conjugation	$x^*[-n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X^*(\frac{1}{z^*})$	$\frac{1}{R_x}$
real part	$\Re e\{x[n]\}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	R_x
imaginary part	$\Im m\{x[n]\}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	R_x
time shifting	$x[n-n_0]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$z^{-n_0}X(z)$	R_x
scaling in $\mathcal{Z}$	$a^n x[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X\left(\frac{z}{a}\right)$	$\lvert a\rvert R_x$
downsampling by N	$x[Nn], N \in \mathbb{N}_0$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{N}\sum_{k=0}^{N-1} X\left(W_N^k z^{\frac{1}{N}}\right) \quad W_N = e^{-\frac{j2\omega}{N}}$	R_x
linearity	$ax_1[n] + bx_2[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$aX_1(z) + bX_2(z)$	$R_x \cap R_y$
time multiplication	$x_1[n]x_2[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{2\pi j}\oint X_1(u)X_2\left(\frac{z}{u}\right)u^{-1}du$	$R_x \cap R_y$
frequency convolution	$x_1[n] * x_2[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$X_1(z)X_2(t)$	$R_x \cap R_y$
delta function	$\delta[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	1	$\forall z$
shifted delta function	$\delta[n-n_0]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	z^{-n_0}	$\forall z$
step	$u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
	$-u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-1}$	$\lvert z\rvert < 1$
ramp	$nu[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
	$n^2u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
	$-n^2u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert < 1$
	$n^3u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert > 1$
	$-n^3u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert < 1$
	$(-1)^n$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z+1}$	$\lvert z\rvert < 1$
exponential	$a^n u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
	$-a^n u[-n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
	$a^{n-1}u[n-1]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
	$na^n u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
	$n^2a^n u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{az(z+a}{(z-a)^3}$	$\lvert z\rvert > \lvert a\rvert$
	$e^{-an}u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{z-e^{-a}}$	$\lvert z\rvert > \lvert e^{-a}\rvert$
exp. interval	$\begin{cases} a^n & n = 0,\ldots,N-1 \\ 0 & \text{otherwise} \end{cases}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
sine	$\sin(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z\sin(\omega_0)}{z^2-2\cos(\omega_0)z+1}$	$\lvert z\rvert > 1$
cosine	$\cos(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z-\cos(\omega_0))}{z^2-2\cos(\omega_0)z+1}$	$\lvert z\rvert > 1$
	$a^n\sin(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{za\sin(\omega_0)}{z^2-2a\cos(\omega_0)z+a^2}$	$\lvert z\rvert > a$
	$a^n\cos(\omega_0 n)\,u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z(z-a\cos(\omega_0))}{z^2-2a\cos(\omega_0)z+a^2}$	$\lvert z\rvert > a$
differentiation in $\mathcal{Z}$	$nx[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$-z\frac{dX(z)}{dz}$	R_x
integration in $\mathcal{Z}$	$\frac{x[n]}{n}$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$-\int_0^z \frac{X(z)}{z}dz$	R_x
	$\frac{\prod_{i=1}^m (n-i+1)}{a^m m!}a^m u[n]$	$\overset{\mathcal{Z}}{\longleftrightarrow}$	$\frac{z}{(z-a)^{m+1}}$	

Note:

$$\frac{z}{z-1} = \frac{1}{1-z^{-1}}$$