UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 2 Examination in Engineering: November 2017

Module Number: IS2401 **Module Name: Linear Algebra & Differential Equations**

[Three hours]

[Answer **all questions**, each question carries **14** marks]

Q1. a) Prove that

i. if $F(\alpha) \neq 0$ then $\frac{1}{F(D)}\{e^{\alpha x}\} = \frac{1}{F(\alpha)} e^{\alpha x}$.

ii. $\frac{1}{1-D}\{x^n\} = \sum_{r=0}^{n} \frac{n!}{(n-r)!} x^{n-r}$

[2 Marks]

b) i. If $t = \log x$ then express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$.

ii. Solve

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = e^{3x} + x^2$$

[5 Marks]

c) What is meant by

i. an ordinary point

ii. a regular singular point of a differential equation of the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = F(x)$$

[2 Marks]

d) i. Show that $x = 0$ is a regular singular point of the differential equation

$$2(x^2 + x^3)\frac{d^2y}{dx^2} - (x - 3x^2)\frac{dy}{dx} + y = 0$$

ii. Solve the above differerntial equation about $x = 0$.

[5 Marks]

Q2. a) i. Define a conservative vector field.

ii. Show that if C is a simple closed curve then $\int_C \mathbf{F}.d\mathbf{r} = 0$, where $\mathbf{F}$ is a conservative vector field.

iii. Show that $\mathbf{F} = (2x\sin y + z)\mathbf{i} + (x^2\cos y - 2y\cos z)\mathbf{j} + (x + y^2\sin z)\mathbf{k}$ is a conservative vector field and hence, find the corresponding scalar potential.

[6 Marks]

b) i. Sketch the region R in the xy plane bounded by $y = x^2, x = 2, y = 1$.

ii. Evaluate $\int \phi \, \mathbf{n} ds$, where $\phi = \frac{3}{8}xyz$, and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$. Where $\mathbf{n}$ is the outward unit normal to the surface S.

[4 Marks]

c) Let $\mathbf{F} = (2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}$. Evaluate

i. $\int_V \nabla . \mathbf{F} \, dv$

ii. $\int_V \nabla \times \mathbf{F} \, dv$,

where V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

[4 Marks]

Q3. a) State

i. The divergence theorem

ii. Stockes' theorem.

[2 Marks]

b) Verify the divergence theorem for $\mathbf{A} = y(x + y)\mathbf{i} + 2y\mathbf{j} + x^2y\mathbf{k}$ taken above the xy-plane bounded by the cylinder $x^2 + y^2 = 4$ and the plane $z = 5$.

[3 Marks]

c) Suppose $\overrightarrow{OA} = a\mathbf{i}$, $\overrightarrow{OB} = a\mathbf{j}$, $\overrightarrow{OC} = a\mathbf{k}$ from three coterminous edges of a cube and S denotes the surface of the cube.

i. Evaluate,

$$\int_S \{(x^3 - yz)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\}. \mathbf{n} ds$$

by expressing it as a volume integral, where $\mathbf{n}$ is the outward unit normal to the surface S.

ii. Verify the results by direct evaluation of the surface integration.

[5 Marks]

d) Let V be the volume bounded by the closed surface S. The vector field $\mathbf{A}$ and the scalar field $\emptyset$ are acting on the surface S. If $\mathbf{n}$ is the outward unit normal to the surface S at the point (x, y, z) and $\mathbf{r}$ is the position vector of the point (x, y, z), prove that

i. $\iiint_V (\nabla \times \mathbf{A}) dv = \iint_S (\mathbf{n} \times \mathbf{A}) \, ds$

ii. $\int_C \emptyset d\mathbf{r} = \iint_S (\mathbf{n} \times \nabla\emptyset) ds = \iint_S \, d\mathbf{S} \times \nabla\emptyset$

[4 Marks]

Q4. a) Let V be a vector space over the field F. Explain what is meant by

i. a subspace of V

ii. a spaning set of V

iii. basis and dimension of V.

iv. sum and direct sum of two subspace U and W of V.

[4 Marks]

b) State whether each of the following is true or false. Justify your answers.

i. $W = \{(x, y); x, y, z \in \mathbb{R},\ x^2 = y^2\}$ is a subspace of $\mathbb{R}^2$ under the usual addition and scalar multiplication.

ii. If $U = \{(x, 0, z); x, z \in \mathbb{R}\}$ and $W = \{(x, y, 0); x, y \in \mathbb{R}\}$, $V = \mathbb{R}^3$ is a direct sum of U and W.

iii. $S = \{(1, 0, 2), (2, -1, 1), (1, -1, -1)\}$ is a basis for $\mathbb{R}^3$.

iv. $T = \left\{\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ spans the vector space $M_{2\times 2}$.

[6 Marks]

c) Let U and W be subspaces of $\mathbb{R}^4$ generated by the following two sets S and T respectively, here,

$$S = \{(1,1,-1,2), (1,2,1,0), (2,1,1,-1), (1,-1,0,-1)\} \text{ and}$$
$$T = \{(1,1,1,2), (0,1,2,-2), (1,2,1,1), (1,2,3,0)\}$$

Find the dimensions of $U + W$ and $U \cap W$.

[4 Marks]

Q5. a) Briefly explain each of the following

i. Linear transformation from vector space V to vector space U.

ii. Kernel and Image of a linear transformation.

[2 Marks]

b) i. Let V and U be two vector spaces and T be a linear transformation from V to U. Show that, if $v_1, v_2, \dots \dots v_n$ span V, then $T(v_1), T(v_2), \dots \dots T(v_{n)})$ span U.

ii. Let $T: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that

$$T(x, y, z, t) = (x, y + z, y - t).$$

Find bases and dimensions of the Kernel and the Image of T.

[6 Marks]

c) i. Find the eigenvalues and eigenvectors of the following matrix A.

$$A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

ii. Diagonalize the above matrix A.

[6 Marks]