UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: July 2017

Module Number: IS3301 **Module Name: Complex Analysis and Mathematical Transforms**

[Three Hours]

[Answer **all questions**, each question carries fourteen marks]

Q1. a) The map $w = \dfrac{az+b}{cz+d}$; $ad - bc \neq 0$ is called bilinear transformation.

i Show that the bilinear transformation w which maps the points z_1, z_2, z_3 of the z plane into the points w_1, w_2, w_3 respectively of the $w-$plane is given by the implicitly by

$$\frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$$

ii What is the bilinear transformation which maps the points $z = -1, 0, 1$ into the points $w = 0, i, 3i$ respectively?

[5 Marks]

b) Explain the followings.

i The function $f(z)$ is continuous at $z = z_0$.

ii The function $f(z)$ is differentiable at $z = z_0$.

iii The function $f(z)$ is entire.

[3 Marks]

c) Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

[2 Marks]

d) i Show that $u(x, y) = e^{-x}(x \sin y - y \cos y)$ is harmonic.

ii Hence, find $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic.

iii Find $f(z)$ in the above d) ii.

[4 Marks]

Q2. a) Evaluate the contour integral $\oint_c f(z)\,dz$ using the parametric representations for c, where $f(z) = \dfrac{z^2 - 1}{z}$ and the curve c is

i the semicircle $z = 2e^{i\theta}$; $(0 \le \theta \le \pi)$.

ii the semicircle $z = 2e^{i\theta}$; $(\pi \le \theta \le 2\pi)$.

iii the circle $z = 2e^{i\theta}$; $(0 \le \theta \le 2\pi)$.

[4 Marks]

b) i State the Cauchy's Integral Formula for derivatives $f^{(n)}(a)$ in the usual notations.

ii Evaluate $\oint_c \frac{z^3+3}{z(z-i)^2}dz$, where c is the contour shown in Figure Q2.

[4 Marks]

c) i State the Cauchy's Residue Theorem.

ii Use the Residue Theorem to compute

$$\oint_{|z|=1} \frac{e^{2/z}}{(1+4z^2)}dz.$$

[3 Marks]

d) Expand $f(z)$, where

$f(z)=\frac{z}{(z-1)(2-z)}$ in a Laurent Series valid for:

i $|z|<1$,

ii $|z-1|>1$.

[3 Marks]

Q3. a) Define the Laplace Transform (L) of the function $f(t)$ for all positive values of t. Show that

i $L[e^{at}f(t)]=F[s-a]$.

ii $L\left[\frac{d}{dt}f(t)\right]=s\,L[f(t)]-f(0)$.

[3 Marks]

b) Find the followings.

i $L[t^2e^t\sin(4t)]$.

ii $L^{-1}\left[\frac{2s}{(s^2+1)^2}\right]$.

[3 Marks]

c) Express the following function $f(t)$ in terms of unit functions and then find its Laplace Transform.

$$f(t)=\begin{cases} t-1\,;\,1<t<2 \\ 3-t\,;\,2<t<3 \end{cases}$$

[3 Marks]

d) An engineering system is modelled by the block diagram in Figure Q3. When $k=2.5$ and $a=0.5$, determine the system response $V_0(t)$ when the input function $V_1(t)$ is a unit step function.

[Hint: If the system has an overall Transfer Function $H(s)$ then $V_0(s)=H(s)V_1(s)\,;\,H(s)=\frac{G(s)}{1+G(s)}$]

[5 Marks]

Q4. a) Consider the Fourier Series for a function $f(t)$ of period 2π;

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$$

Where, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt \quad ; n=1,2,3,\ldots \quad , \quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt \quad ; n=1,2,3,\ldots$

Define the periodic function $f(t)$ for the Figure Q4. a). i and Q4. a). ii and then obtain the Fourier Series of $f(t)$ for each of them.

[6 Marks]

b) Obtain the formulas for Fourier Series and Fourier Coefficients when the given series is

i odd.

ii even.

[3 Marks]

c) Find the Fourier Series for $f(x)$ on $[-3,3]$, where

$$f(x)=\begin{cases}-1 & ; \ -3\le x<0\\ 1 & ; \ 0\le x\le 3\end{cases}$$

[Hint: In the usual notations, Fourier Series for the function $f(x)$ of general period is, $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c}+b_n\sin\frac{n\pi x}{c}\right)$.

Where,

$a_0=\frac{1}{c}\int_0^{2c}f(x)\,dx \quad , \quad a_n=\frac{1}{c}\int_0^{2c}f(x)\cos\frac{n\pi x}{c}\,dx \quad , \quad b_n=\frac{1}{c}\int_0^{2c}f(x)\sin\frac{n\pi x}{c}\,dx$]

[5 Marks]

Q5.

a) Prove that for $0<x<\pi$,

$$x(\pi-x)=\frac{\pi^2}{6}-\left[\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 6x}{3^2}+\ldots\right].$$

Hence, deduce $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

[6 Marks]

b) If $F(S)$ be the Fourier Transform of the function $f(x)$ in the usual notation as $F(S)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{ist}\,dt$, then find the Fourier Sine Transform of $f(x)=e^{-ax}$.

[3 Marks]

c) i Define the $Z-$Transforms for a causal sequence $x(n)$.

ii If it is given that the sequence $x(n)=u(n)$ in the usual notation, find the $Z-$ Transform of $x(n)$ with the region of convergence.

[2 Marks]

d) Given the sequences,

$x_1(n)=3\delta(n)+2\delta(n-1)$

$x_2(n)=2\delta(n)-\delta(n-1)$.

Find the $Z-$Transforms of their Convolution.

[3 Marks]

Figure Q2: Contour

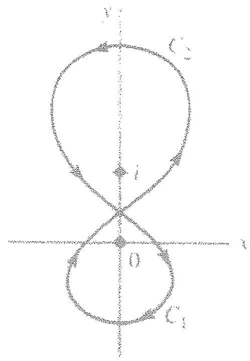

Figure Q3: Block Diagram for an Engineering System

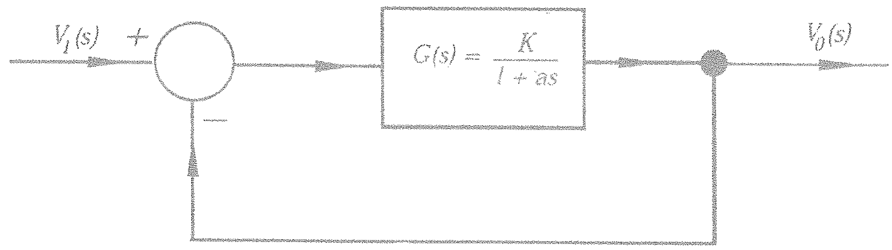

Figure Q4. a). i

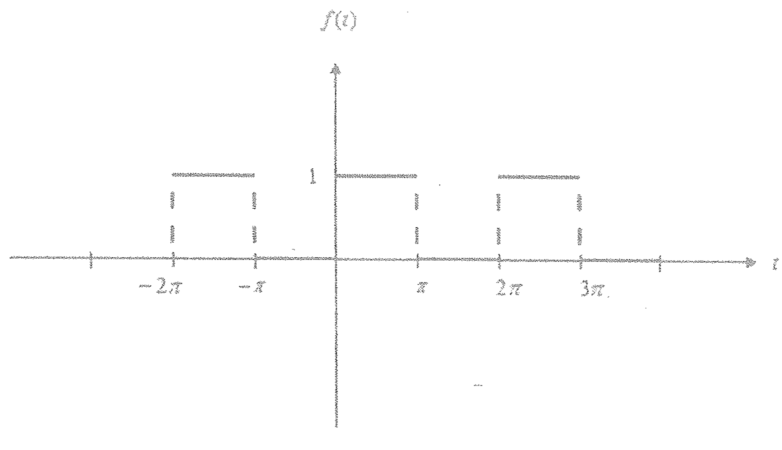

Figure Q4. a). ii

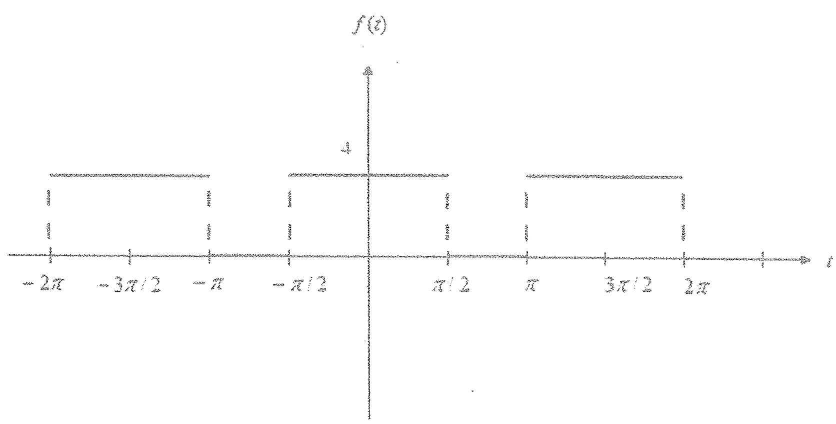

Table Q3: Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n e^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	e^{-cs}
27.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	28.	$u_c(t)g(t)$	$e^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$e^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^{\infty}F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$				$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

	Signal, $x[n]$	Z-transform, $X(z)$	ROC
1	$\delta[n]$	1	all z
2	$\delta[n-n_0]$	z^{-n_0}	$z \neq 0$
3	$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
4	$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
5	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
6	$-nu[-n-1]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert < 1$
7	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
8	$-n^2u[-n-1]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert < 1$
9	$n^3u[n]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\lvert z\rvert > 1$
10	$-n^3u[-n-1]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\lvert z\rvert < 1$
11	$a^nu[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
12	$-a^nu[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
13	$na^nu[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
14	$-na^nu[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
15	$n^2a^nu[n]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\lvert z\rvert > \lvert a\rvert$