UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 7 Examination in Engineering: March 2021

Module Number: EE 7209 **Module Name: Digital Signal Processing**

[Three Hours]

[Answer all questions, each question carries 10 marks]

[Notations and symbols have their usual meaning unless otherwise stated]

[If necessary, you may use the provided information in Table I and Table II]

Q1 a) The impulse response $h[n]$ of a discrete-time linear time-invariant (LTI) system $\boldsymbol{H}$ and an input signal $x[n]$ are shown in Figure Q1. Both can be assumed to be zero beyond the regions shown.

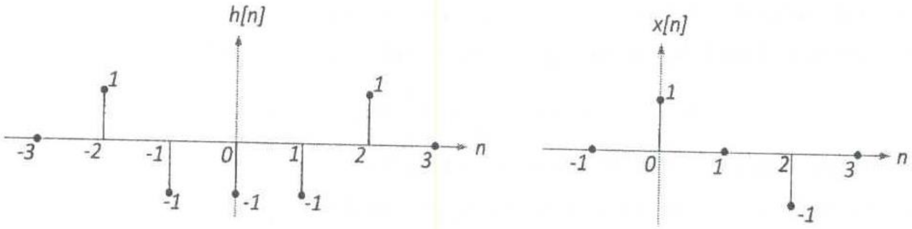

Figure Q1.

i. State and explain if the system is causal or non-causal. [1 mark]

ii. State and briefly explain if the system is bounded-input bounded-output (BIBO) stable. [1 mark]

iii. Derive the output $y[n] = \boldsymbol{H}\{x[n]\}$. [2 marks]

b) Let $y[n]$ be a sequence that is generated from a sequence $x[n]$ as follows:

$$y[n] = \sum_{k=-\infty}^{n} kx[k]$$

I. Show that $y[n]$ satisfies the time-varying difference equation $y[n] - y[n-1] = nx[n]$. [1 mark]

II. Show that

$$Y[z] = \left(\frac{-z^2}{z-1}\right)\frac{dX[z]}{dz}$$

where $X[z]$ and $Y[z]$ are the z-transforms of $x[n]$ and $y[n]$ respectively.

Hint: $nx[n] \overset{Z}{\leftrightarrow} -Z\frac{dX[z]}{dz}$ [2 marks]

III. Use some of the above results to find the z-transform of

$$y[n] = \sum_{k=0}^{n} k\left(\frac{1}{3}\right)^{k} \qquad n \geq 0.$$

[3 marks]

Q2 A causal discrete-time LTI system $\boldsymbol{H}$ is formed by three causal discrete-time LTI systems H_1, H_2, and H_3 as shown in Figure Q2.

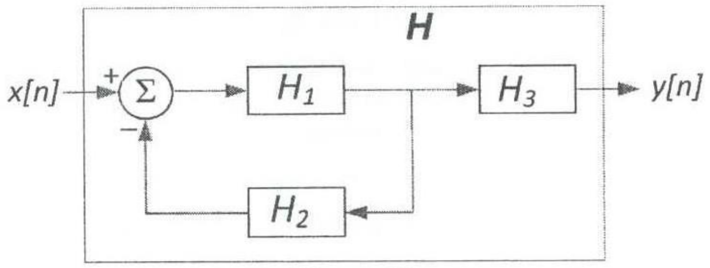

Figure Q2.

The input-output relation for H_1 is given by $y_1[n] - 4y_1[n-1] = x_1[n]$. The impulse responses $h_2[n]$ and $h_3[n]$ are given by $h_2[n] = \alpha\delta[n]$, and

$$h_3[n] = \delta[n] - \left(\frac{1}{2} + \beta\right)\delta[n-1] + \frac{1}{2}\beta\delta[n-2]$$

respectively where $\alpha, \beta \in \mathbb{R}$ are real constants.

a) Obtain the z-transforms of $h_1[n], h_2[n]$ and $h_3[n]$. [3 marks]

b) Show that

$$\boldsymbol{H}[z] = \left(\frac{H_1[z]}{1 + H_1[z]H_2[z]}\right) H_3[z].$$

[2 marks]

c) Determine the values of the real constants α and β so that $\boldsymbol{H}$ is both causal and stable and also the inverse system $\boldsymbol{G}[z] = {}^{1}/_{\boldsymbol{H}[z]}$ is both causal and stable.

[5 marks]

Q3 a) I. State the convolution theorem for Fourier transforms.

[1 mark]

II. By applying the above theorem, determine the convolution of the sequences $x_1[n] = x_2[n] = \{1,\underset{\uparrow}{1},1\}$. Time origin ($n = 0$) is indicated by the symbol $\uparrow$.

Note: No mark will be given if tabulation or graphical methods are used.

[4 marks]

b) Let $x[n]$ be a length-N discrete-time signal given by $x[n] = 1 - \delta[n-(N-1)]$.

I. Show that

$$x[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & n = N-1 \end{cases}$$

[1 mark]

II. Find the N-point Discrete Fourier Transform (DFT) of $x[n]$. [4 marks]

Q4 a) The Fast Fourier Transform (FFT) is a widely used algorithm for spectral analysis of signals. The radix-2 decimation-in-time FFT algorithm rearranges the Discrete Fourier Transform (DFT) equation into two parts.

I. Briefly mention the advantage of using the FFT in signal processing?

[1 mark]

II. Split the N-point DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0,1,2,\dots,N-1$$

into **two** summations such that

$$X[k] = G[k] + W_N^k H[k]$$

where $G[k]$ denote the sum over the even-numbered discrete-time indices $n = [0,2,4,\dots,N-2]$ and $H[k]$ denote the sum over the odd-numbered discrete-time indices $n = [1,3,5,\dots,N-1]$.

Note that the twiddle factor $W_N = e^{-\frac{j2\pi}{N}}$ and $W_N^{2nk} = W_{N/2}^{kn}$.

[3 marks]

III. Figure Q4 illustrates the computation of an $N = 8$ point DFT, $X[k]$, by using the sum of the outputs of two DFTs. Fill in the blanks in Figure Q4 with correct notations or values based on the results obtained in Q4 a) II.

Note: Blanks are indicated as ⬚.

Note: Figure Q4 should be attached to your answer script.

[3 marks]

b) Let $x[n]$ be a sequence of length N with

$$x[n] = -x\left[n + \frac{N}{2}\right], \qquad n = 0,1,2,\dots,\frac{N}{2} - 1$$

where N is an even integer.

Show that the N-point DFT of $x[n]$ has only odd harmonics, that is, when k is even $X[k] = 0$.

[3 marks]

Q5 a) Sketch the Direct form-I structure for the realization of the following discrete-time system.

$$y[n] = \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + x[n] + \frac{1}{3}x[n-1].$$

[2 marks]

b) A parallel-form realization of an infinite-impulse response (IIR) filter can be obtained by performing a partial-fraction expansion of the system function $H[z]$. Determine and sketch the parallel-form structure of the system

$$H[z] = \frac{4 + \frac{9}{4}z^{-1} - \frac{1}{4}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}.$$

[4 marks]

c) The impulse invariance method is applied to design an IIR filter having a unit sample response $h[n]$ which is the sampled version of the impulse response of the continuous analog filter. That is

$$h[n] \equiv h_a[nT] \qquad n = 0,1,2,\ldots$$

where T is the sampling interval.

The system function of an analog filter is given as

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

where $\{p_k\}$ are the poles of the analog filter and $\{c_k\}$ are the coefficients in the partial-fraction expansion. Consequently,

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t} \quad t \geq 0.$$

I. With the help of the above information, prove that the system function of the resulting digital IIR filter is equal to

$$H[z] = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}.$$

[2 marks]

II. Use the impulse invariance method to design a digital IIR filter from an analog prototype that has a system function

$$H_a(s) = \frac{1}{s^2 + 3s + 2}$$

assuming that the sampling frequency is 5 Hz. [2 marks]

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Note: Figure Q4 should be attached to the answer script.

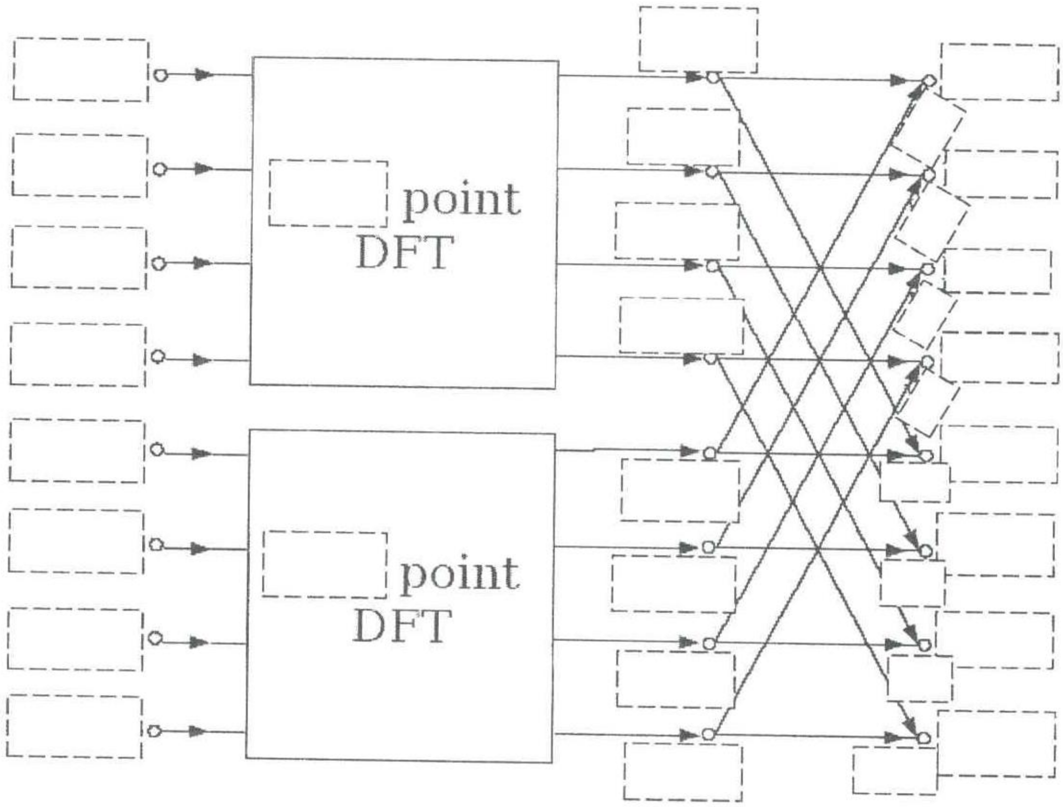

Figure Q4.

Table I: Some common z-Transform pairs

Signal, $x[n]$	z-Transform, $X[z]$	Region of convergence
$\delta[n]$	1	All z
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\lvert z \rvert > \lvert a \rvert$
$-a^n u[-n-1]$	$\frac{1}{1 - az^{-1}}$	$\lvert z \rvert < \lvert a \rvert$
$\cos[\omega_0 n]\, u[n]$	$\frac{1 - z^{-1}\cos(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}}$	$\lvert z \rvert > 1$
$\sin[\omega_0 n]\, u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1 - 2z^{-1}\cos(\omega_0) + z^{-2}}$	$\lvert z \rvert > 1$

Note: The unit sample sequence and the unit step signal are denoted as $\delta[n]$ and $u[n]$ respectively.

Table II: Frequency analysis of discrete-time signals

Periodic signals		Aperiodic signals	
Synthesis equation	$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$	Synthesis equation	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega) e^{j\omega n}\, d\omega$
Analysis equation	$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$	Analysis equation	$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$