UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 5 Examination in Engineering: December 2020

Module Number: IS5306 **Module Name: Numerical Methods**

[Three hours]

[Answer **all questions,** each question carries 12 marks]

Q1. a) By considering one practical application, briefly explain the importance of the use of numerical methods for solving scientific or engineering problems.

[2 Marks]

b) Clearly mentioning the assumptions, use Taylor's expansion to prove the Newton-Raphson formula,

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}.$$

[2 Marks]

c) A loan of 'A' rupees is repaid by making n equal monthly payments of M rupees, starting a month after the loan is made. It can be shown that if the monthly interest rate is r, then

$$Ar = M\left(1 - \frac{1}{(1+r)^n}\right).$$

A car loan of Rs. 2,500,000.00 was repaid in 60 monthly payments of Rs. 50,000.00. Use Newton-Raphson method to find the monthly interest rate (%) with an accuracy of 0.001.

[8 Marks]

Q2. a) The Lagrangian interpolating polynomial of degree n, passes through $n+1$ data points (x_0,y_0), (x_1,y_1),............, (x_{n-1},y_{n-1}), (x_n,y_n) is defined as $P_n(x) = \sum_{i=0}^{n} y_i L_i(x)$.

Where, $L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$ and n is a positive integer.

Show that $L_i(x_j) = 1$ when $i = j$ and
$L_i(x_j) = 0$ when $i \neq j$.

[2 Marks]

b) i.) Write down the general form of the Newton's divided difference interpolation polynomial.

[1 Mark]

ii.) The velocity distribution of a fluid near a flat surface is given below.

Distance, x (cm)	0.1	0.3	0.5	0.7	0.9
Velocity, V (cm/s)	0.75	1.80	2.75	3.45	4.00

Where x is the distance from the surface and V is the velocity. Using the Newton's divided difference method of interpolation with 4th order polynomial, obtain the velocity at $x = 0.6$ cm.

[4 Marks]

c) Given the system of equations

$$\begin{aligned} x_1 + 5x_2 + 3x_3 &= 28 \\ 3x_1 + 7x_2 + 13x_3 &= 76 \\ 12x_1 + 3x_2 - 5x_3 &= 1 \end{aligned}$$

with an initial guess of $x_1^{(0)} = 1$, $x_2^{(0)} = 0$ and $x_3^{(0)} = 1$. Solve the system by using Gauss Seidel method.

[5 Marks]

Q3. a) The first level of processing what we see involves detecting edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges for different values of a, where a is a constant, derivatives of functions such as,

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x < 0 \end{cases}$$ need to be found.

i.) Calculate the functions 1st derivative $f'(x)$ at $x = 0.1$ for $a = 0.15$, by using the central difference approximation. Use a step size of $h = 0.05$.

ii.) Calculate the functions 2nd derivative $f''(x)$ at $x = 0.1$ for $a = 0.15$, by using the central difference approximation. Use a step size of $h = 0.05$.

iii.) Calculate the absolute relative true errors (%).

[6 Marks]

b) i.) Use composite Simpson's rule with six subintervals to find an approximate value for the integral,

$$\int_1^{1.8} \ln\left(1 + x^2\right)\, dx,$$ correct to four decimal places.

ii.) Use your answer in part (i) to deduce an approximate value for

$$\int_1^{1.8} \ln\left(e^{3x}\sqrt{1 + x^2}\right)\, dx.$$

[6 Marks]

Q4. a) Briefly explain the following by giving an example for each.

i.) Ordinary differential equations (ODE)
ii.) Partial differential equations (PDE)
iii.) Initial value problem (IVP)
iv.) Boundary value problem (BVP)

[3 Marks]

b) Solve, $\frac{dy}{dx} = x - y$, for $x = 0.1$ by using Picard's successive approximation method correct to four decimal places, given that $y = 1$ when $x = 0$.

[3 Marks]

c) A polluted lake has an initial concentration of a bacteria of 10^7 parts/m^3, while the acceptable level is only 10^5 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by,

$$\frac{dC}{dt} + 0.05Ct = 0, \qquad C(0) = 10^7.$$

Using the Runge-Kutta 4th order method, find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

[6 Marks]

Q5. a) Classify the following equations as linear or non-linear, and state their order.

i.) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$

ii.) $\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w\frac{\partial w}{\partial x} = 0$

iii.) $2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial t} + 3\frac{\partial^2 f}{\partial t^2} + 4\frac{\partial f}{\partial x} + \cos(2t) = 0$

[3 Marks]

b) Classify the following partial differential equations as hyperbolic, parabolic, or elliptic.

i.) $8\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0$

ii.) $\alpha\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

iii.) $\frac{\partial^2 u}{\partial x^2} + x^2\frac{\partial^2 u}{\partial y^2} = 0, \quad x \neq 0$

[3 Marks]

c) Solve the heat equation,

$$\frac{\partial^2 u(x,t)}{\partial x^2} - 2\frac{\partial u(x,t)}{\partial t} = 0 \quad \text{for } 0 \le x \le 1 \text{ and } 0 \le t \le 0.05$$

with the initial conditions,

$$u(x,0) = f(x) = x - x^2$$

and the boundary conditions,

$$u(0,t) = 0$$
$$u(1,t) = t.$$

Use, $h = 0.25$ and $k = 0.025$, where h and k are step sizes along x and t axes respectively.

[6 Marks]