UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: July 2016

Module Number: IS3301 **Module Name: Complex Analysis and Mathematical Transforms**

[Three Hours]

[Answer **all questions**, each question carries fourteen marks]

Q1.

a) Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ and u and v are real valued functions. Suppose further that $f'(z)$ exists.

i State the Cauchy-Riemann equations.

ii Let the function $f(z)$ be analytic in a domain D, and suppose that f satisfies $\text{Re}(f(z)) = \text{Im}(f(z))$ for $\forall z \in D$. Show that the function f is constant in D.

[4 Marks]

b) i Define those "a function is analytic at the point z_0" and "a function is entire" for a given function $f(z)$.

ii Show that there is no entire function f such that $f'(z) = xy^2$ for $\forall z \in C$.

[4 Marks]

c) A real valued function $\phi(x, y)$ is said to be harmonic in a given domain D if ϕ has continuous partial derivatives up to the second order in D and satisfying the Laplace's equation.

i Show that the function $u(x, y) = e^{-x} \cos y + xy$ is harmonic.

ii Then find a harmonic conjugate of the harmonic function $u(x, y)$.

[6 Marks]

Q2.

a) If $f(z)$ is a single-valued continuous function in some region R in the complex plane, then define the integral of $f(z)$ along a path c in R.

Use Figure Q2 to obtain the following complex integrals.

i $\int_{c_1} z\, dz$

ii $\int_{c_2} z\, dz$

[3 Marks]

b) State the Cauchy's Integral Formula in the usual notation. Hence evaluate the following integrals.

i $\displaystyle\oint_{|z|=4} \frac{\cos z}{z^2 - 6z + 5}\, dz$

ii $\displaystyle\oint_{|z+i|=4} \frac{1}{z-3}\, dz$

[4 Marks]

c) Find the Taylor series expansion of the following functions at $z = 0$.

i $f(z) = e^z$

ii $f(z) = \dfrac{1}{1-z}$

[3.0 Marks]

d) Find the Laurent Series of the function $f(z) = \frac{1}{z^2 - 2z}$ centered at the origin in the domain;

i $|z| < 2,$

ii $|z| > 2.$

[4 Marks]

Q3.

a) Determine the nature of all singularities of the following functions.

i $f(z) = \dfrac{1}{z^2 \sin z}$

ii $f(z) = \dfrac{z}{e^{z^2} - 1}$

[5 Marks]

b) Consider the complex valued function defined by,

$$f(z) = \frac{3z^3 + 2}{(z-1)(z^2+9)}.$$

i Find the singular points of the function f.

ii Find the residues of the function f at the above singular points.

iii Hence, evaluate the following integral.

$$\oint_{|z|=4} \frac{3z^3 + 2}{(z-1)(z^2+9)}\, dz$$

[9 Marks]

Q4.

a) Let $F(t)$ be a function of t; $t > 0$. Then define the Laplace transform of $F(t)$. Then, show, in the usual notation, that

i $$L\{X(t)\} = \frac{1}{s^2}\ ;\ s > 0$$

ii $$L\{e^{at}\} = \frac{1}{s-a}\ ;\ s > a$$

[6 Marks]

b) Find the Laplace transform of the following functions.

i $$F(t) = \begin{cases} 5 & ;\ 0 < t < 3 \\ 0 & ;\ t > 3 \end{cases}$$

iii $$F(t) = 1 + \cos 2t$$

[4 Marks]

c) Use Laplace transform to solve the following Initial Value Problem.

$$Ri + L\frac{di}{dt} = v\ ;\ R = 4W,\ L = 2H,\ V(t) = 10\sin 5t$$

(You may use that $L\{F'(t)\} = sf(s) - F(0)$ in the usual notation).

[4 Marks]

Q5.

a) The Fourier series of a function $f(x)$ with period $2L$ is,

$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$. Where the Fourier coefficients a_n and b_n are,

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}dx,\quad b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}dx\ ;\ c \text{ is any real number.}$$

Let $f(x)$ be a function such that $f(x) = \begin{cases} 0 & ;\ -5 < x < 0 \\ 3 & ;\ 0 < x < 5 \end{cases}$.

i Find the Fourier coefficients a_0, a_n and b_n.

ii Find the corresponding Fourier series.

[7 Marks]

b) Find the Fourier transform of the following functions.

i $$f(x) = \begin{cases} 1 & ;\ |x| < b \\ 0 & ;\ |x| > b \end{cases};$$

where b is any real number.

ii $$f(x) = \begin{cases} 1 & ;\ -2 < x < 2 \\ 0 & ;\ otherwise \end{cases}$$

[4 Marks]

c) The $Z-$ transform of a finite sequence $\{f(k)\}$ is denoted as $Z[\{f(k)\}]$ and defined as $Z[\{f(k)\}] = F(Z) = \sum_{k=0}^{N} f(k)z^{-k} = \sum_{k=0}^{N} \delta[k-n_0]z^{-k} = z^{-n_0}$.

Thus,

$\delta[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}$.

i Use the $Z-$transform to find the zeros of the sequence; $h[n] = \delta[n] + \frac{1}{6}\delta[n-1] - \frac{1}{6}\delta[n-2]$.

ii Convolve the following two sequences.
$x[n] = 2\delta[n] - 3\delta[n-2] + 4\delta[n-3]$
$p[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$.

[3 Marks]

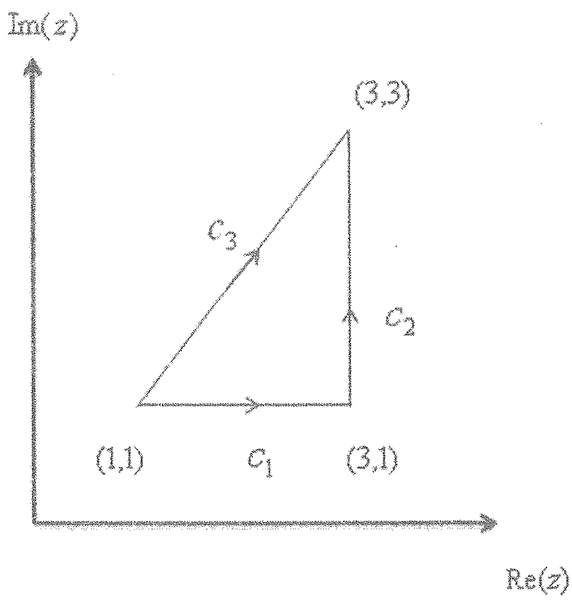

Figure Q2

Table of Laplace Transforms

$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. e^{at}	$\frac{1}{s-a}$
3. $t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6. $t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12. $\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17. $\sinh(at)$	$\frac{a}{s^2-a^2}$	18. $\cosh(at)$	$\frac{s}{s^2-a^2}$
19. $e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20. $e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21. $e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22. $e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23. $t^n e^{at}, \quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25. $u_c(t)=u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26. $\delta(t-c)$ Dirac Delta Function	e^{-cs}
27. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	28. $u_c(t)g(t)$	$e^{-cs}\mathcal{L}\{g(t+c)\}$
29. $e^{ct}f(t)$	$F(s-c)$	30. $t^n f(t), \quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31. $\frac{1}{t}f(t)$	$\int_s^{\infty} F(u)\,du$	32. $\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34. $f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$
35. $f'(t)$	$sF(s)-f(0)$	36. $f''(t)$	$s^2F(s)-sf(0)-f'(0)$

37. $f^{(n)}(t)$ $\qquad s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$