UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 3 Examination in Engineering: July 2016

Module Number: ME 3303 **Module Name: Modelling of Dynamic Systems**

[Three Hours]

[Answer **all questions**, each question carries twelve marks]

A partial table of Laplace transformation pairs is given in page 7. You may make additional assumptions where necessary, but clearly state them in your answers. Symbols stated herein denote standard parameters.

Q1. a) Use Laplace transformation to find the solutions for the following ordinary differential equations.

i) $$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+10y=\sin 2t;\ y(0)=1,\ \dot{y}(0)=1$$

ii) $$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+10y=-\delta(t-4\pi);\ y(0)=0,\ \dot{y}(0)=1$$

$\delta(t)$: *Dirac delta function*

[7.0 Marks]

b) Consider the *LCR* electrical network shown in Figure Q1 (b).

i) Find the transfer function $G(s) = V_o(s)/V_i(s)$.

ii) Find the output response; $v_o(t)$ in terms of homogeneous response and exogenous response for a unit-step input voltage, $v_i(t)$.

iii) Identify the steady state response and comment on the behavior of the output response.

[5.0 Marks]

Note: Current across a capacitor; $i=C\frac{dv}{dt}$.

Q2. a) Determine the output, $Y(s)$ of the block diagram shown in Figure Q2 (a).

[3.0 Marks]

b) A toy train shown in Figure Q2 (b) consists of an engine and a car. The train travels only in one direction. It is required to apply control to the train so that it has a smooth start-up and stop, along with a constant-speed ride. The mass of the engine and the car will be represented by M_1 and M_2, respectively. The engine is coupled to the car by a spring having stiffness k. F represents the force applied by the engine, and the Greek letter μ represents the coefficient of rolling friction.

i) Show the system in a Free Body Diagram.

ii) Obtain the transfer function of the toy train.

Q2 is continued to next page...

Assumptions:

- Let the output of the system be the velocity of the engine.
- The train starts at zero speed.
- When the train is moving at a velocity v, the force acting due to rolling friction is given by $\mu M v g$.

[6.0 Marks]

c) Briefly explain the importance of modeling and simulation when developing a mechatronic product?

[1.5 Marks]

d) Outline a numerical technique commonly used in time-domain simulations?

[1.5 Marks]

Q3. a) A height adjustable stabilized platform built with a shock-absorber system [$G(s)$] and a single gain [K] feedback path is shown in Figure Q3 (c-i). In the system, the control input [$u(t)$] is adjusted dynamically in order to make the response [$y(t)$] to reach the reference height [$r(t)$]. The equivalent closed loop block diagram for the entire system is shown in the Figure Q3 (c-ii). Parameters k and b stand for the spring and damper coefficients of the shock-absorber system, respectively.

i) Show that the open loop transfer function of the shock-absorber system is represented by, $G(s) = \dfrac{\eta}{s^2 + 2\sigma s + \rho}$.

Where $\sigma = b/2m$, $\rho = k/m$, $\eta = 1/m$.

ii) Obtain the closed loop transfer function of the stabilized platform shown in Figure Q3 (c-ii).

iii) Find the poles of the characteristic equation and show that the platform achieves its marginal stability at $K = -\dfrac{\rho}{\eta}$.

iv) Show that the platform undergoes a critically damped stage at $K = \dfrac{\sigma^2 - \rho}{\eta}$.

v) Comment on the behavior of the system when; $-\dfrac{\rho}{\eta} < K < \dfrac{\sigma^2 - \rho}{\eta}$.

vi) Given that the m=0.1 kg, b=1 Ns/m, k=2 N/m for the stabilized platform; Prove that the system shows an oscillatory stable response when the feedback gain [K] reaches to 4.

[8.0 Marks]

b) Calculate the natural un-damped frequency, peak overshoot and settling time for the platform described in part (vi).

[2.0 Marks]

c) What would be the minimum value of the damper coefficient in order to maintain the settling time below 0.5 s?

Q3 is continued to next page...

Hint:
For under-damped generic second order systems;

The transfer function, $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, peak overshoot; $PO = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$,

settling time; $T_s \approx \dfrac{4.6}{\zeta\omega_n}$ with usual notations.

[2.0 Marks]

Q4. a) The state space representation of a dynamic system is given by the state equation; $\dot{X} = AX + BU$ and the output equation; Y=CX+DU with usual notations. Show that the system transfer function can be represented by,
$G(s)=C(sI-A)^{-1}B+D$.

[2.0 Marks]

b) The schematic diagram of a robot arm is shown in Figure Q4 (b). Here, the joint motor of the robot arm is excited with a variable DC voltage $v(t)$. It is required to determine how the arm position; $\theta(t)$ changes when the voltage; $v(t)$ changes. List of other relevant parameters are given in the Table 1.

Total inertia on the motor shaft is equal to the sum of motor inertia and the load inertia, when coupled through n: 1 reduction gear. Therefore, the equivalent inertia on the motor shaft is given by $J_{eq}=J_m + (1/n^2) J_l$. Similarly, equivalent viscous friction coefficient on the motor shaft is $b_{eq}=b_m + (1/n^2) b_l$. Torque on the motor shaft overcomes the friction (velocity dependent), and drives the inertia (acceleration dependent) as described by; $\tau_m(t) = b_{eq}\dot{\theta}_m(t) + J_{eq}\ddot{\theta}_m(t)$.

Table 1: Parameters of the Robot arm system

$v(t)$: armature voltage [V]	$i(t)$: armature current [A]
J_m: motor inertia [Kgm^2]	k_T: torque constant [Nm/A]
n: gear ratio	τ_l: load shaft torque [Kgm^2]
L: armature inductance [Vs/A]	$v_B(t)$: back electromotive force [V]
b_m: motor viscous damping constant [Nms/rad]	$\dot{\theta}_m(t)$: motor shaft speed [rad/s]
J_l: arm inertia [kgm^2]	R: armature resistance [Ω]
k_B: motor back emf constant [Vs/rad]	τ_m: motor shaft torque [Nm]
$\theta_m(t)$: motor shaft position [rad]	b_l: viscous damping constant of the robot arm [Nms/rad]
$\theta(t)$: robot arm position [rad]	

i) Obtain a third order differential equation to describe the robot arm joint position, as a function of supply voltage (input variable).

ii) Obtain the state space model of the system (state equation, output equation and A, B, C, D matrices) with usual notations.

Q4 is continued to next page...

iii) Draw the block diagram for the state space system.

[6.0 Marks]

c) The robot arm system described in part b, can be numerically represented by following state and output equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -9 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u],$$

$$[\theta] = [1\ 0\ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Determine the eigenvalues and hence comment on the stability of the robot arm.

[4.0 Marks]

Q5. a) Lyapunov first method (Indirect method) determines the stability nature of the equilibrium state (at the origin) of a nonlinear autonomous dynamic system. The method generalizes the concept of energy; V for a conservative system in mechanics, where a well-known result states that an equilibrium point is stable if the energy is minimum. Thus V is a positive function which has $\dot{V}$ negative in the neighborhood of a stable equilibrium point. Apply Lyapunov first method (stability based on energy) to determine the stability of the spring mass system shown in Figure Q5 (a).

[3.0 Marks]

b) Lyapunov second method (direct method) defines a precise stability criterion eliminating drawbacks of the first method. The theorem states that, "A linear time invariant system, $\dot{X} = AX$ is asymptotically stable if and only if for any positive definite matrix Q there exists a positive definite symmetric solution P to the Lyapunov equation, $A^T P + PA = -Q$". Use Lyapunov second method to determine the stability of the state matrix; A=$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ using the Lyapunov equation with $Q = I$ (Identity Matrix).

[3.0 Marks]

c) ~~b)~~ The governing equation of a simple pendulum of length L and total mass M with an oscillated angle of θ is given by,

$$\ddot{\theta} + \frac{3C}{ML^2}\dot{\theta} + \frac{3g}{2L}\sin\theta = 0.$$

Where C is the viscous frictional torque due to air resistance and bearing resistance acting on the pendulum and, g is the acceleration due to gravity.

i) By using the state space representation, obtain the dynamic system model of the nonlinear system.

ii) Apply Lyapunov's linearization method to obtain equilibrium solutions for the dynamic system.

iii) Find all equilibrium solutions and investigate the stability.

[6.0 Marks]

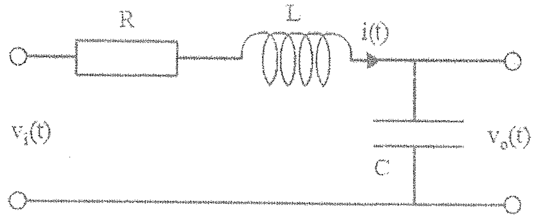

Figure Q1 (b): LCR Circuit

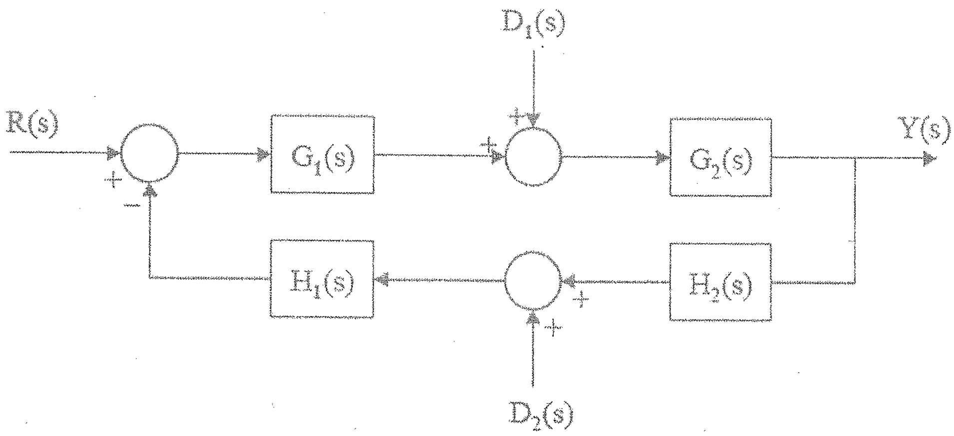

Figure Q2(a): Block Diagram

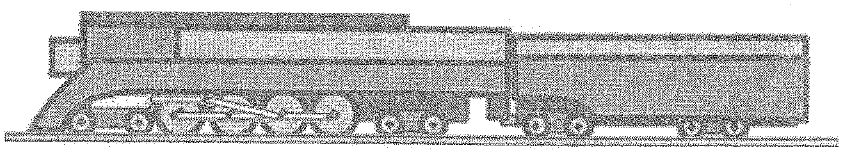

Figure Q2(b): Toy Train

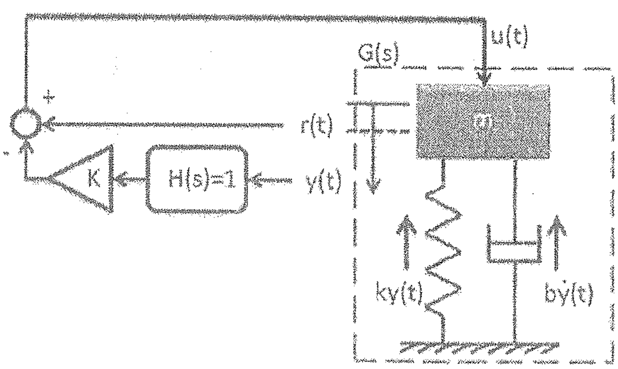

Figure Q3(c-i): Height Adjustable Stabilized Platform

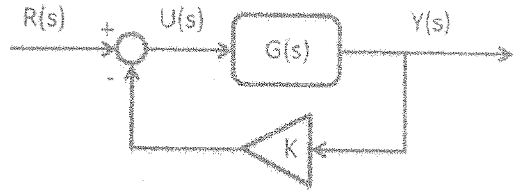

Figure Q3(c-ii): Equivalent Block Diagram of the Platform

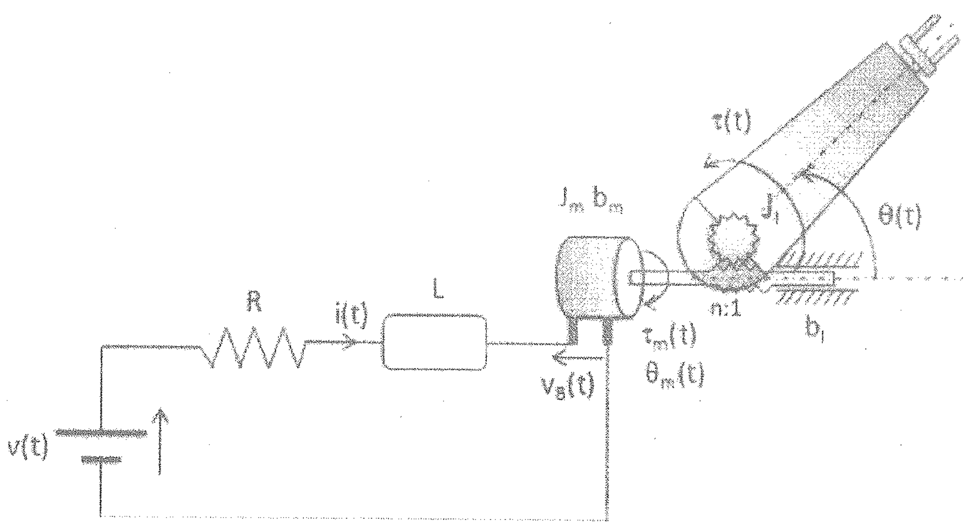

Figure Q4(b): Robot Arm and the Joint Motor

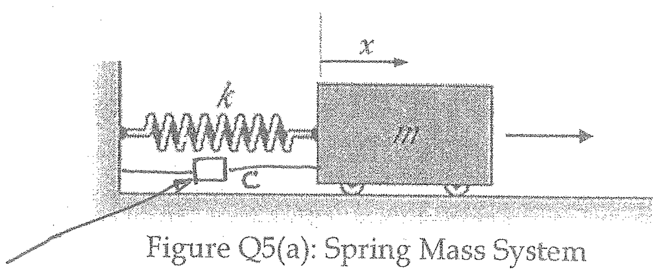

Figure Q5(a): Spring Mass System

Damping Coefficient.

Table of Laplace transform pairs

$f(t)$	$F(s)$
step	$\frac{1}{s}$
ramp, t	$\frac{1}{s^2}$
impulse	1
dirac delta function, $\delta(t-c); c \geq 0$	e^{-cs}
$u(t-a)$	$\frac{e^{-as}}{s}$
$u(t-a)\ g(t-a)$	$e^{-as}\,G(s)$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{(s+a)}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\frac{1}{(n-1)!}\, t^{n-1}\, e^{-at}$	$\frac{1}{(s+a)^n}$
$\sin \omega t$	$\frac{\omega}{(s^2+\omega^2)}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{\omega_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\omega_0 \sqrt{1-\zeta^2 t})$	$\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$
$1-\frac{\omega_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\omega_0 \sqrt{1-\zeta^2} t+\phi)$ *Where*, $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{\omega_0^2}{s(s^2+2\zeta\omega_0 s+\omega_0^2)}$