UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 5 Examination in Engineering: July 2016

Module Number: IS5301 **Module Name: Numerical Methods**

[Three hours]

[Answer **all questions**, each question carries 14 marks]

Q1.

a) By considering one practical application, briefly explain the importance of the use of numerical methods for solving scientific and/or engineering problems.

[1.5 Marks]

b) i.) Clearly mentioning the assumptions, prove the Newton-Raphson formula,

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ using Taylor's expansion. [2.0 Marks]

ii.) Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, temperature can be determined. For a 10K3A Betatherm thermistor, the relationship between the resistance (R) of the thermistor and the temperature (T) is given by,

$$\frac{1}{T} = 1.13\times10^{-3} + 2.34\times10^{-4}\ln(R) + 8.78\times10^{-8}\{\ln(R)\}^3$$

where T is in Kelvin and R is in ohms.

Use the Newton-Raphson method to find the resistance R at 18.99°C, correct to 2 decimal places by assuming the initial guess of the root R_0 = 15000Ω. Find the absolute relative approximate error (%) at the end of each iteration.

[8.0 Marks]

iii.) Show that the order of convergence of the 'Newton-Raphson method' is quadratic.

[2.5 Marks]

Q2.

a) The Lagrangian interpolating polynomial of degree n that passes through $n+1$ data points (x_0,y_0), (x_1,y_1),............, (x_{n-1},y_{n-1}), (x_n,y_n) is defined as $P_n(x) = \sum_{i=0}^{n} y_i L_i(x)$.

where, $L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \dfrac{x - x_j}{x_i - x_j}$.

Show that $L_i(x_j) = 1$ when $i = j$ and

$L_i(x_j) = 0$ when $i \neq j$. [2.0 Marks]

b) A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15cm x 10cm rectangular plate (x and y directions respectively). The centres of the holes in the plate describe the path of the arm needs to take, and the hole centres are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in following table.

x, (cm)	2.0	4.5	5.5	8.0	11.5
y, (cm)	8.0	7.5	7.0	6.4	5.5

i.) If the laser is traversing from $x = 2.0$ to $x = 4.5$ in a linear path, find the value of y at $x = 4.0$ using Newton's divided difference method of interpolation with a first order polynomial.

[2.0 Marks]

ii.) If the laser is traversing in a biquadratic path, find the value of y at $x = 4.0$ using Newton's divided difference method of interpolation with a 4th order polynomial.

[4.0 Marks]

c) Given the system of equations

$$x_1 + 5x_2 + 3x_3 = 29$$
$$3x_1 + 7x_2 + 13x_3 = 79$$
$$12x_1 + 3x_2 - 5x_3 = 13$$

with an initial guess of $x_1^{(0)} = 1$, $x_2^{(0)} = 0$ and $x_3^{(0)} = 1$,

i.) find whether the system has a strictly diagonally dominant coefficient matrix.

ii.) if so, solve the system and if not, re-arrange the system and solve it by using Gauss Seidel method.

[6.0 Marks]

Q3.

a) The first level of processing what we see involves detecting edges or positions of transitions from dark to bright or bright to dark points in images. These points usually coincide with boundaries of objects. To model the edges, derivatives of functions such as,

$$f(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ e^{ax} - 1, & x \leq 0 \end{cases}$$ need to be found.

i.) Calculate the functions 1st derivative $f'(x)$ at $x = 0.1$ for $a = 0.12$, by using the central difference approximation. Use a step size of h = 0.05.

ii.) Calculate the functions 2nd derivative $f''(x)$ at $x = 0.1$ for $a = 0.12$, by using the central difference approximation. Use a step size of h = 0.05.

iii.) Calculate the absolute relative true errors (%).

[7.0 Marks]

b) The distance covered by a rocket in metres from $t = 7.5$s to $t = 35$s is given by,

$$x = \int_{7.5}^{35} f(t)\, dt$$

where, $f(t) = \left(2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)$.

i) Use the composite Simpson's rule with five subintervals to find the approximate value of x.

ii) If the function $f(t)$ shown above is known to have a fourth derivative with the property that, $\left|f^{(iv)}(t)\right| \le 0.012$ for $7.5 \le t \le 35$, determine how many subintervals are required so that the composite Simpson's rule used to approximate x, incurs an error less than 0.02.

[7.0 Marks]

Q4.

a) Briefly explain the following giving an example for each item.

i.) Ordinary differential equations (ODE)
ii.) Partial differential equations (PDE)
iii.) Initial value problem (IVP)
iv.) Boundary value problem (BVP)

[3.0 Marks]

b) A polluted lake has an initial concentration of a bacteria of 10^7 parts/m^3, while the acceptable level is only 10^5 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by,

$$\frac{dC}{dt} + 0.08Ct = 0, \qquad C(0) = 10^7$$

Using the Runge-Kutta 4th order method, find the concentration of the pollutant after 7 weeks. Take a step size of 3.5 weeks.

[8.0 Marks]

c) Using the following ordinary differential equation with given initial conditions and step size of h, show that the error of 4th order Runge-Kutta method is of order 5 [$O(h^5)$].

$$\frac{dy}{dx} = y, \qquad y(0) = 1$$

[3.0 Marks]

Q5.

a) Classify the following equations as linear or non-linear, and state their order.

i.) $\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$

ii.) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$

iii.) $\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w\frac{\partial w}{\partial x} = 0$

iv.) $2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial t} + 3\frac{\partial^2 f}{\partial t^2} + 4\frac{\partial f}{\partial x} + \cos(2t) = 0$

[4.0 Marks]

b) Solve the heat equation,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x \leq 2 \text{ and } 0 < t \leq 0.6$$

with the initial conditions,

$u(x,0) = f(x) = x\,(2-x)$

and the boundary conditions,

$u(0,t) = 0$

$u(2,t) = t^2$.

Use, $h = 0.5$ and $k = 0.2$, where h and k are step sizes along x and t axes respectively.

[7.0 Marks]

c) Briefly explain the procedure of finite difference solution technique for solving Laplace equation $\left(\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0\right)$ highlighting any differences with the solution technique used in part (b).

[3.0 Marks]