UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 2 Examination in Engineering: December 2016

Module Number: IS2401 **Module Name: Linear Algebra and Differential Equations**

[Three hours]

[Answer **all questions**, each question carries fourteen marks]

Q1. a) Solve the following linear differential equations.

i. $$\frac{dy}{dx} + 3x^2y = x^2 + e^{-x^3}$$

ii. $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}; \quad y(0) = 1, \quad \left.\frac{dy}{dx}\right|_{x=0} = -1$$

[6 Marks]

b) Use Frobenious method to obtain two linearly independent solutions about $x = 0$ of the following differential equation.

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x^2 + 1)y = 0.$$

Show that the infinity is a regular singular point to the following differential equation and hence, determine its solution about infinity.

$$2x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + \left(\frac{1}{x^2} + 1\right)y = 0$$

[8 Marks]

Q2. a) i. What is meant by a conservative vector field.

ii. If $\mathbf{A} = \nabla\phi$, show that A is a conservative vector field.
Hence, deduce that the work done in moving a particle in this vector field from P to Q is $\phi(Q) - \phi(P)$.

[3 Marks]

b) If $\mathbf{A} = (y + 1)\mathbf{i} - z\,\mathbf{j} + xz\ \mathbf{k}$, find the work done in moving a particle once around a circle C. Where C is the circle with the radius 2 and the centre at (0,0,1) in the in the $z = 1$ plane.

[3Marks]

c) Find the surface integral of the vector field $\mathbf{F} = z\mathbf{i} - x\,\mathbf{j} + 2yz\ \mathbf{k}$ outward through the surface of the sphere $x^2 + y^2 + z^2 = 4$ above the xy plane.

[4 Marks]

d) If $\mathbf{A} = (x + 2y^2)\mathbf{i} + 2y(3 - z)\,\mathbf{j} - 2z\,\mathbf{k}$, compute the volume integral

$$\int_V \nabla.\mathbf{A}dV,$$

where V is the volume of a cylinder $x^2 + y^2 = 1,\ z = 0$ and $z = 5$.

[4 Marks]

Q3. a) i. State the Green's theorem on a plane.

ii. Use Greens theorem to evaluate

$$\oint_C (2x^2 + xy)dx + (x - 2y^2)dy,$$

where C is the closed curve of the region bounded by $y^2 = 8x$ and $y = x^2$

[3 Marks]

b) Use Stokes' theorem to evaluate

$$\iint_S \nabla \times \mathbf{A}\, ds.$$

Where, $\mathbf{A} = y\mathbf{i} - 2z\,\mathbf{j} - x\,\mathbf{k}$ where S is the open hemisphere $x^2 + y^2 + (z-2)^2 = 4$.

[3 Marks]

c) i. State the divergence theorem.

If S is the surface cut by the paraboloid $x^2 + y^2 = 1 - z$ in the first octant and z varies from 0 to 1, use divergence theorem to evaluate

$$\iint_S (x^2 z\mathbf{i} + y^2\,\mathbf{j} - xz^2\,\mathbf{k}).\mathbf{n}dS$$

ii. If S is any closed surface enclosing a volume V and $\mathbf{n}$ is the unit outward drawn normal to the surface$S$, show that

$$\iint_S \phi\mathbf{n}dS = \iiint_V \nabla\phi dV$$

Hence, deduce that

$$\iint_S r^5\mathbf{n}dS = \iiint_V 5r^3\mathbf{r}dV$$

[8 Marks]

Q4. a) Define the followings:

i. Subspace of a vector space.

ii. Kernel and Image under a linear transformation

[2 Marks]

b) State whether each of the following is true or false. Justify your answers.

i. $W = \{x, y, z: x^2 + y^2 + z^2 = 1\}$ is a subspace of $\mathbb{R}^3$ under usual addition and scalar multiplication.

ii. If U and W are subspaces of a vector space V then $U \cup W$ is also a subspace of V.

iii. $S = \{(1, 0, -1, 2), (2, -1, 1, 0)\}$ is a linearly independent set but not spans $\mathbb{R}^4$

iv. $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(x, y, z) = (x + y, x - z)$ is a linear transformation.

[8 Marks]

c) Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 4 \\ -1 & 1 & -1 \end{bmatrix}$$

be the matrix representation of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^4$.

Determine the rank, nullity and a basis for the null space of the linear transformation T.

[4 Marks]

Q5. a) Let U and W be subspaces of $\mathbb{R}^4$ generated by the sets $\{(1,-1,0,2), (2,0,1,1)(1,1,1,-1)\}$ and $\{(0,2,1,-3), (1,-1,1,3), (1,1,2,0)\}$ respectively.
Find dimensions of $U + W$ and $U \cap W$.

[4 Marks]

b) If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the eigenvalues of a square matrix A of order n, show that:

i. the matrix $A - kI$ has eigenvalues $(\lambda_1 - k), (\lambda_2 - k), \ldots\ldots, (\lambda_n - k)$, where k is a constant.

ii. the matrix A^2 has eigenvalues $\lambda_2^2, \lambda_1^2, \ldots\ldots, \lambda_n^2$

[2 Marks]

c) Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ -2 & 4 & 0 \end{bmatrix}$$

Is A diagonalizable? Justify your answer.

Determine the eigenvalues of $A - I$ and A^2.

[8 Marks]