UNIVERSITY OF RUHUNA
BACHELOR OF SCIENCE SPECIAL DEGREE(LEVEL I) SEMESTER I
EXAMINATION APRIL - 2021

PHY4024 - Classical Mechanics and Special Relativity **Time: 03 Hours**

Answer Six Questions Only

Mass of the electron = 0.511 MeV/c²

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1. An inextensible string of length a passes through a hole in the middle of a large frictionless table. A point mass m at one end of the string, is placed at distance r from the hole and is free to move on the table. Another point mass of $2m$ hangs from the other end of the string and constrained to move vertically. Assume that the second object hangs always beneath the hole throughout the motion.

   (a) Obtain the Lagrangian for the system in terms of r and angle θ that the mass m makes with respect to a fixed axis on the horizontal table.

   (05 Marks)

   (b) Find the equations of motion.

   (05 Marks)

   (c) Identify the conserved quantities in the system and write down an expression for each quantity as a function of r and θ.

   (03 Marks)

   (d) Find the distance (r_0) from the hole to the mass m when the hanging mass stays stationary.

   (05 Marks)

   (e) If the hanging mass is pulled down slightly from the stationary point show that the angular frequency of oscillations is equal to $(\frac{m}{l})^{\frac{1}{3}}(2g)^{\frac{2}{3}}$.

   (07 Marks)

2. Halley's comet with mass m moves under a central potential $V(r) = -\frac{k}{r}$ in an elliptical orbit.

   (a) Write down the Lagrangian of the system using plane polar coordinates (r, θ).

   (03 Marks)

   (b) Obtain the Lagrange's equation for r.

   (03 Marks)

(c) Identify the cyclic coordinate and show that the angular momentum (l) is conserved for the system.

(02 Marks)

(d) Show that the orbit equation can be written as,
$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{dV(\frac{1}{u})}{du}$, where $u = \frac{1}{r}$.

(07 Marks)

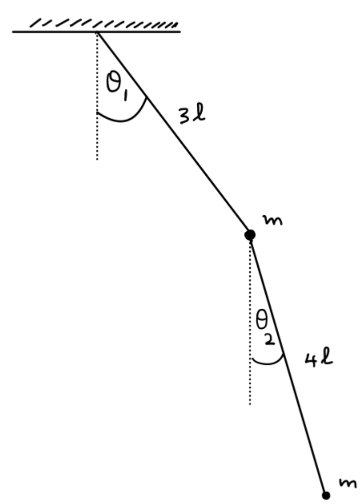
(e) Show that the orbit of the comet can be written as $r(\theta) = \frac{C}{1+\ \epsilon\ cos\ \Phi}$, where $C = \frac{l^2}{mk}$ and ϵ is the eccentricity of the orbit.

(05 Marks)

(f) Halley's comet follows a very eccentric orbit with $\epsilon = 0.967$. At the closest approach, the comet is 0.59 AU away from the sun. What is the comet's greatest distance from the Sun in terms of AU?

(05 Marks)

3. A simple pendulum of length $3l$ and mass m is hanging on a ceiling. A second pendulum of length $4l$ with similar mass is attached to the end of the first pendulum.



(a) Write down the kinetic energy (T) and potential energy (V) of the system.

(05 Marks)

(b) If the system performs small oscillations, show that T and V can be written as,
$T = 2mgl^2 \begin{pmatrix} 3C & 2C \\ 2C & 4E \end{pmatrix}$ and $V = 2mgl \begin{pmatrix} C & 0 \\ 0 & E \end{pmatrix}$; C and E are positive Integers.

(05 Marks)

(c) Find the normal frequencies.

(07 Marks)

(d) Find the normal modes.

(05 Marks)

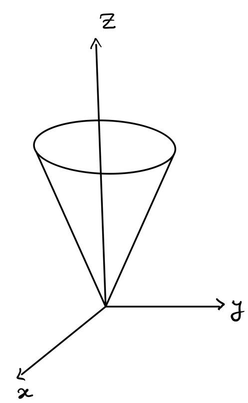
(e) Locate a point on the second pendulum where there is no horizontal displacement.

(03 Marks)

4. (a) Consider the Legendre transformation $H(q_i.p_i,t) = \dot{q}_i p_i - L(q_i,\dot{q}_i,t)$. H and L are the Hamiltonian and the Lagrangian, respectively. Show that $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ and $\dot{q}_i = \frac{\partial H}{\partial p_i}$, where q_i and p_i are the generalized coordinates and momenta, respectively.

(05 Marks)

(b) Consider a mass m which is constrained to move on a frictionless surface of a vertical cone where $\rho = cz$ in cylindrical polar coordinates (ρ, ϕ, z). The cone is fixed with its tip on the ground and kept at a uniform gravitational field.



i. Show that the Lagrangian of the system can be written as $L = \frac{1}{2}m[(c^2+1)\dot{z}^2 + (cz\dot{\phi})^2 - mgz]$.

(03 Marks)

ii. Find the Hamiltonian of the system.

(07 Marks)

iii. Derive Hamilton's equations. Hence identify cyclic coordinates and conserved quantities.

(05 Marks)

iv. Derive an expression for the energy (E) and find the relationship between E and H. Is energy conserved?

(02 Marks)

v. Show that the mass can never fall into the bottom of the cone as long as $p_\phi \neq 0$.

(03 Marks)

5. (a) The transformation between two sets of canonical coordinates (Q_i, P_i) and (q_i, p_i) are related by,
$p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{dF_1}{dt}$, where $F_1 = F_1(q_i, Q_i, t)$. Show that the generating functions of the form $F_3(p_i, Q_i, t)$ transform (q_i, p_i) to (Q_i, P_i) variables according to
$q_i = -\frac{\partial F_3}{\partial p_i}$, $P_i = -\frac{\partial F_3}{\partial Q_i}$, $K = H + \frac{\partial F_3}{\partial t}$.

(05 Marks)

(b) i. Write down the fundamental Poisson brackets for p_i and q_i where p_i and q_i are canonical variables.

(03 Marks)

ii. The transformation equations between two sets are given as
$Q = \log\left(\frac{1}{q}\sin p\right)$
$P = q\cot p$. Assuming that both p and q are canonical variables, show that P and Q are canonical variables.

(07 Marks)

iii. Show that the function that generates the transformation between these two sets of canonical variables is $F_3 = e^{-Q}\cos p$.

(08 Marks)

iv. Suggest a systematic method to obtain generating functions.

(02 Marks)

6. (a) Consider electrons accelerated to a total energy of 20.0 MeV in a 3 km long Linear Accelerator.

i. Find the Lorentz factor (γ) for the electron.

(04 Marks)

ii. What is the electron's speed at the given energy?

(04 Marks)

iii. What is the relativistic momentum of the electron at this speed?

(03 Marks)

iv. What is the length of the accelerator in the electron's frame of reference when they are moving at their highest speed?

(04 Marks)

(b) An electron and a positron annihilate with equal and opposite momentum $p = 1.55$ GeV/c. The collision produces a new particle called J/ψ in the following reaction;

$$e^- + e^+ \rightarrow J/\psi$$

What is the mass of this new particle?

(10 Marks)

7. (a) What is meant by Doppler effect?

(03 Marks)

(b) Consider a source, which propagates waves around the source with frequency f_s. Two observers John and Kate stand left and right sides of the source, respectively. John and Kate are stationary observers.

i. Discuss about the frequencies observed by John and Kate when the source moves towards John.

(06 Marks)

ii. What is the reason for this observation?

(04 Marks)

iii. What happens if both observes and source move at the same velocity?

(02 Marks)

(c) An ambulance truck emits a sound with a frequency of 800 Hz. The speed of sound in air is 343 m s^{-1}.

i. What is the frequency detected by a stationary observer if the ambulance truck is moving 30 m s^{-1} towards the observer?

(05 Marks)

ii. What frequency will be detected if the ambulance truck is moving 30 m s^{-1} away from the observer?

(05 Marks)

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